Introduction to Uncertainty

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Measurements and uncertainties

- **Uncertainty**
- **Types of uncertainty**
- **Standard uncertainty**
- **Confidence Intervals**
- **Expanded Uncertainty**
- **Examples**

Introduction

- Measurement shapes scientific theories, characterizes improvements in manufacturing processes and promotes efficient commerce.
- Inherent in measurement is uncertainty, and students in science and engineering need to identify and quantify uncertainties in the measurements they make.
- ‘uncertainty’ is a measure of dispersion .

Type A and Type B uncertainties

CATEGORIES OF UNCERTAINTY

Type A uncertainty

- A type A uncertainty is one which can be found by **statistical analysis** .

1. **Calculating the Mean**
2. **The scatter around the mean**

- In a situation where a quantity changes with time, the rate of change, commonly called **slope**, is of interest.
- This again will be a Type A uncertainty.

Type B uncertainty

- A Type B uncertainty is due to the presence of systematic error and may be determined by from a calibration report, data book a device manual.
- This uncertainty may include the resolution of the device or zero errors, least counts etc.
- Obtaining this information is the primary purpose of calibrating a device.
- Reading the report several times will give exactly the same result!
- If Z is the best estimate of the correction due to systematic error, then measurand y can be predicted as:

$$\mathbf{y = y_{mean} \pm Z}$$

Degrees of freedom

- Degrees of freedom, ν is the minimal number of values which should be specified to determine all the data points.
- If we have N-1 values for an observation and the mean for N observations, then we can calculate the Nth value, which means that we have N-1 degrees of freedom.
- Degrees of freedom for type B uncertainties is taken to be ∞.

Type A

DIFFERENCE BETWEEN STANDARD DEVIATION, STANDARD ERROR AND STANDARD ERROR IN MEAN

- The standard deviation *s, is an index that measures how* closely the individual data points cluster around the mean, given by: $\mathbf{s=\sum(y_i-y_{mean})^2/n)^{1/2}}$

- Standard Error σ is evaluated by repeating the measurement process many times. It gives the measure of how much an individual value y_i varies from the true value Y**.** $\mathbf{\sigma= (\sum y_i-Y)^2/n)^{1/2}}$

- Standard error in mean σ_{mean} tells us how much the individual means y_{mean} of a sample vary from the true value Y*, this is given by:*
  $\mathbf{\sigma_m = \sigma / \sqrt{n}}$

- Standard error in mean is a primary measure of uncertainty, given the name ‘standard uncertainty’ and denoted by **u(y)** in the rest of the context.

Standard deviation is the deviation of the individual values y_i from the sample mean y_{mean}

$$s^2 = \frac{\sum d_i^2}{n}$$

Standard error is the deviation of the individual values y_i from the true value Y

$$\sigma^2 = \sum \frac{e_i^2}{n}$$

Standard error in mean is the deviation of the each sample's mean from the true value

$$\sigma_m^2 = \frac{\sigma^2}{n}$$

In terms of standard deviation, σ_m can be expressed as

$$\sigma_m^2 = \frac{1}{n-1} \quad s^2$$

Type B Standard uncertainty due to the limit of resolution

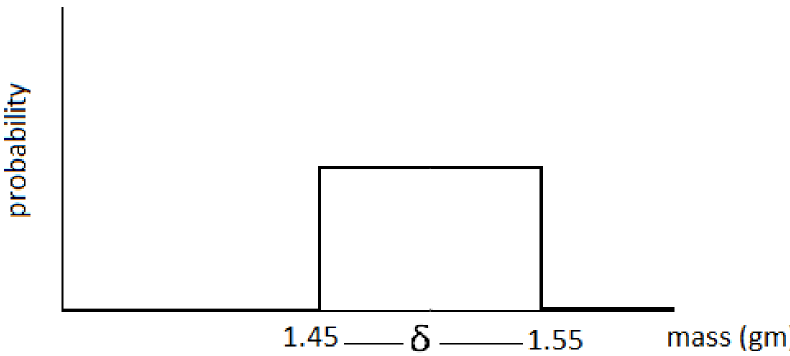
- The distribution of the errors that make up a Type B uncertainty is uniform since there are no statistical treatment available of repeating measurements.
- *In this case the* standard uncertainty is expressed as

  $u(Z) = \delta/\sqrt{12}$.
- The limit of resolution is represented by the symbol ***δ.***

[2] *Refer to Section 8.3 for the derivation of u(Z) expression.*

Example

- The distribution of the errors that make up a Type B uncertainty is uniform, constituting a rectangular probability distribution (in contrast to Gaussian for Type A) and is shown in figure to the left.
- To understand this rectangular distribution, we can take the example of a digital balance, which measures the mass and displays only two decimal digits. If the display is 1.5, then the actual reading may be anywhere, and with uniform probability, within the (approximate) interval 1.45 gm to 1.55 gm.
- We accordingly have $\delta = 0.1$ *gm*



Combining uncertainties

- We report the combined uncertainty formed from the combination of the Type A and Type B components as follows:
- $\mathbf{u^2(y) = u^2(y_{mean}) + u^2(Z)}$
- If $y = f(x_1, x_2, \ldots, x_n)$ *the standard uncertainty, u(y), in y resulting from standard uncertainties* $u(x_1), u(x_2), \ldots, u(x_n)$ *in the input* quantities is calculated using the equation

$$u^2(y) = \left(\frac{\partial y}{\partial x_1}\right)^2 u^2(x_1) + \left(\frac{\partial y}{\partial x_2}\right)^2 u^2(x_2) + \left(\frac{\partial y}{\partial x_3}\right)^2 u^2(x_3) + \cdots + \left(\frac{\partial y}{\partial x_n}\right)^2 u^2(x_n).$$

Linearly varying data

- If we have data for a measurand, y that shows a positive slope with respect to the independent variable x, then we use the data by fitting a straight line to the points using the least-squares condition.
- The slope of the line will equal the rate of change of y. We write the equation describing the straight line as **y=mx+c.**
- So our chief concern will be to find the best estimates of the slope m and intercept c, using the least square curve fit.
- Whenever a straight line is fitted by least-squares, degrees of freedom is 2 less than the number of original values, so we have

  $\boldsymbol{v = n - 2}$.

Expression for slope and intercept

- We first define the quantity *D, as* follows:

$$D = n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2.$$

- In terms of D, the slope m and intercept c are defined as:

$$m = \frac{\sum_{i=1}^{n} y_i \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i y_i \sum_{i=1}^{n} x_i}{D}$$

$$c = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{D}.$$

- The residuals, ε_i *(i = 1, 2, . . ., n), are calculated as:*

  $\varepsilon_i = y_i - c - mx_i$

Uncertainties in slope and intercept

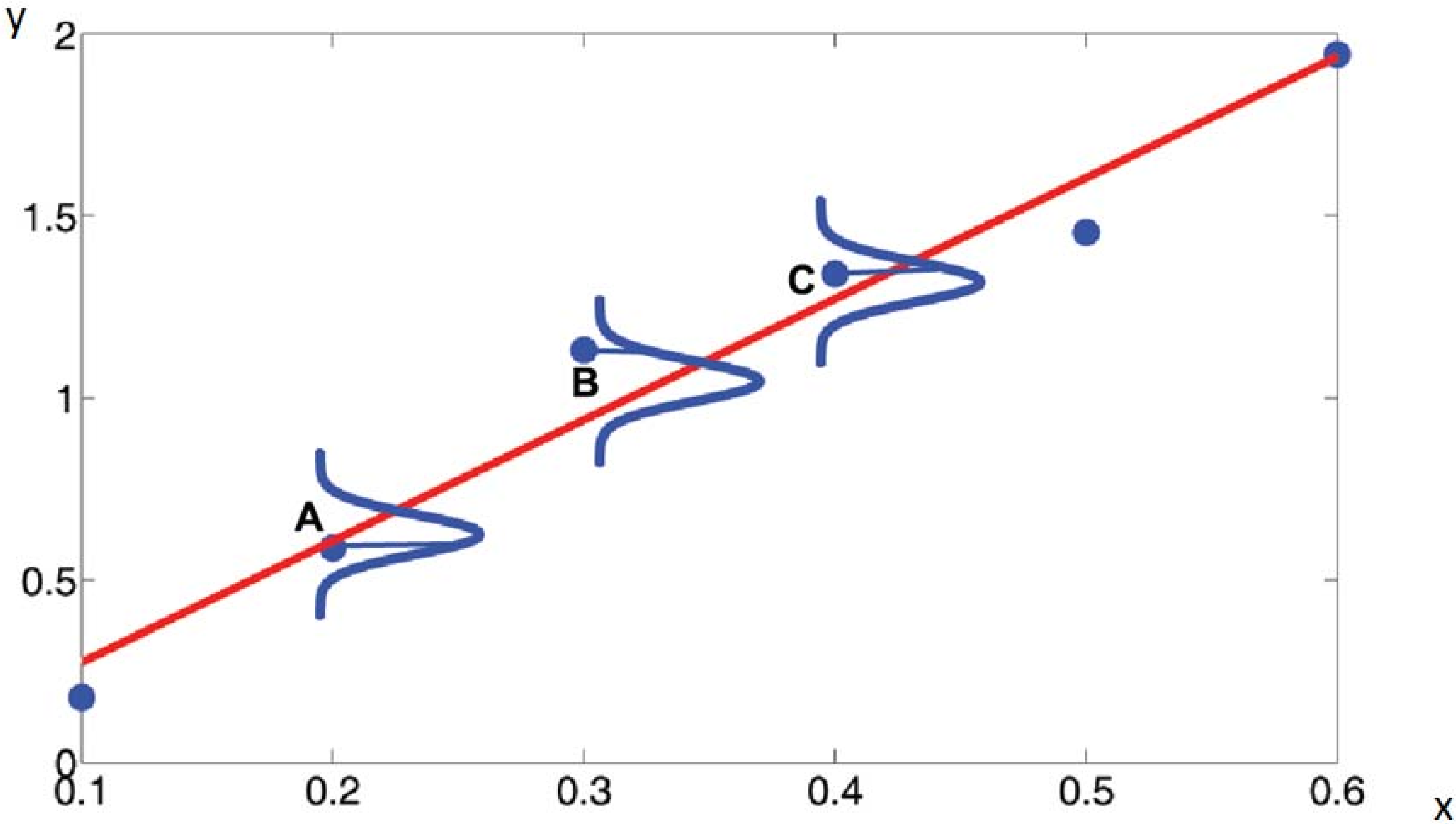
- The root-mean-square value, s is given as

$$s = \sqrt{\frac{\sum_{i=1}^{n} \epsilon_i^2}{n-2}}$$

- The standard uncertainties of slope and intercept, s_m and s_c are derived as:

$$s_m = s\sqrt{\frac{\sum_{i=1}^{n} x_i^2}{D}} \qquad s_c = s\sqrt{\frac{n}{D}}$$

Uncertainty in the estimate of slope

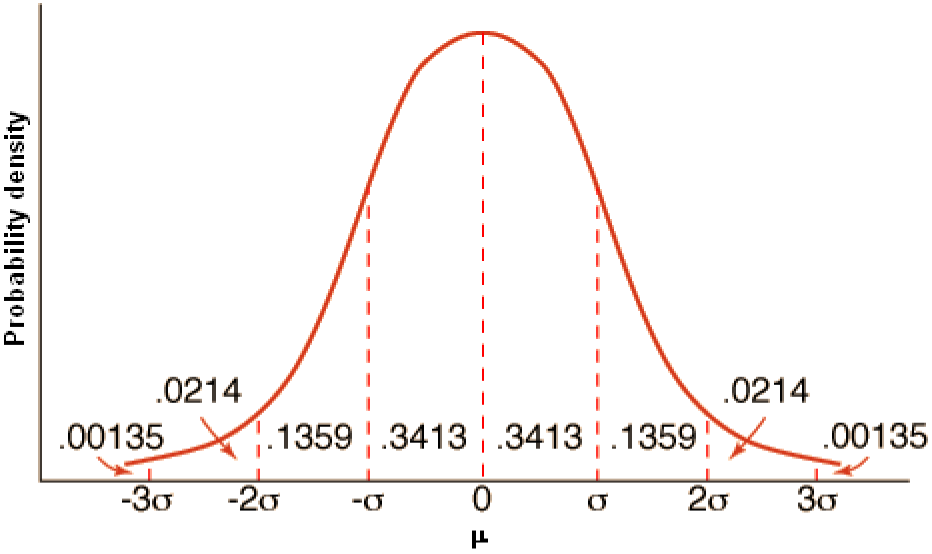


Normal / Gaussian Distribution

- The Normal distribution is the most important and most widely used distribution in statistics.
- Normal distributions have a mean and a standard error.
- Normal distributions are symmetric around their mean.
- The area under the normal curve is equal to 1.0.
- Normal distributions are denser in the center and less dense in the tails.
- Normal distributions are defined by two parameters, the population mean μ and the standard error σ.
- 68% of the area of a normal distribution is within one standard error of the mean.
- Approximately 95% of the area of a normal distribution is within 1.96 times standard error of the mean.

- If the number of events is very large, then the Gaussian distribution function may be used to describe physical events centered around the true mean of a population.
- Standard deviation of the Gaussian distribution, depicted as an envelope to the probabilities, is a natural measure of the Type A standard uncertainty created by random errors.

A plot of a normal distribution (or bell curve). Each band has a width of one standard error.



t-distribution:

- In probability and statistics, the *t-distribution* is used for a normally distributed population when the degrees of freedom or the sample size is small, and the standard error has to be estimated from the sample data.
- t-distribution is specified completely by the number of degrees of freedom.
- It is a good approximation to the distribution of the means of randomly drawn samples from a fixed population.

Confidence Intervals

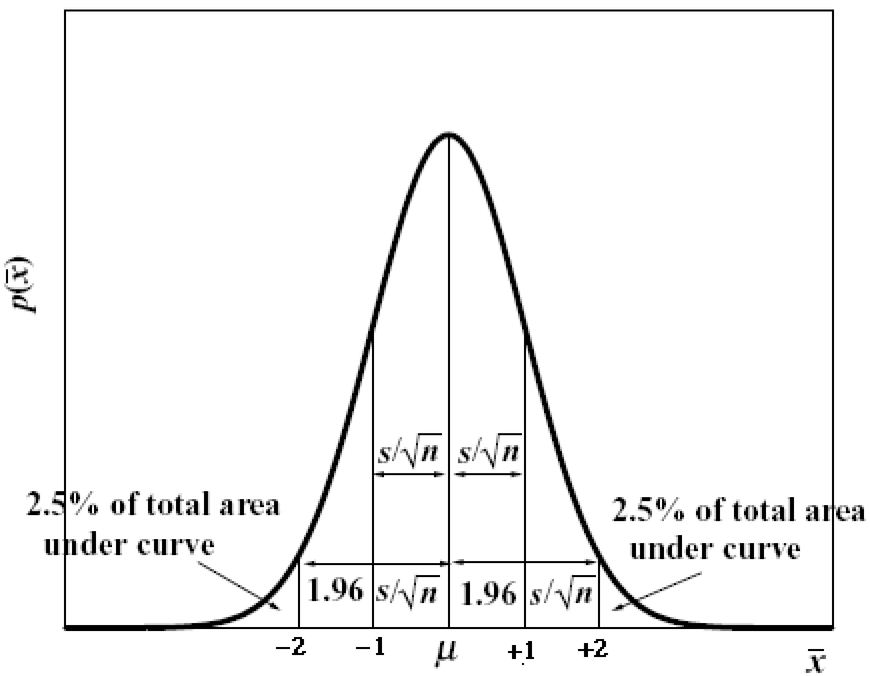
- **A *confidence interval* gives an estimated range of values which is likely to include an unknown population parameter, the estimated range being calculated from a given set of sample data.**
- **The selection of a confidence level for an interval determines the probability that the confidence interval produced will contain the true parameter value. Common choices for the confidence level are 90%, 95% and 99%. These levels correspond to percentages of the area of the normal density or t-distribution curve. For example, a 95% confidence interval covers 95% of the distribution-- the probability of observing a value outside of this area is less than 5%.**
- **In theory, we can construct intervals of any level of confidence from 0 to 100%. The shorter the confidence interval, the less likely it is to contain the quantity being estimated. The longer the interval, the more likely it is to contain the quantity being estimated, but the penalty we pay for our increased confidence is a wider CI, so that we are less sure of the exact value.**
- **95% has been found to be a convenient level for conducting scientific research, so it is used almost universally.**

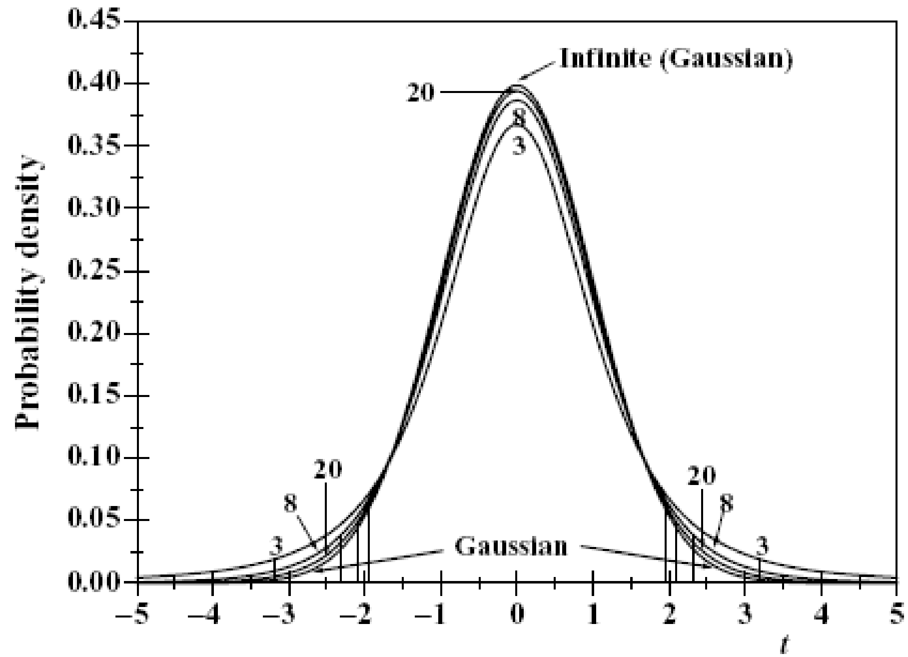
Use of Standard error (SE) for determining Confidence Intervals

- The SE is an index of the variability of the means that would be expected if the study were exactly replicated a large number of times.
- By itself, SE doesn't convey much useful information. What is often required is a measure of uncertainty that defines an interval about the measurement result y within which the value of the measurand *y* can be confidently asserted to lie.
- So the main function of SE's is to help construct CI's, which can supplement statistical significance testing and indicate the range within which the true mean or difference between means may be found.
- If we look at the distribution in the normal curve, we'll find that 95% of the area falls between -1.96 and +1.96 SEs.

Guassian distribution for infinite v.

t-distributions for v = 3, 8, 20 and∞.





t-distribution with lesser degrees of freedom have more area near the tails

Confidence interval calculation for a t-distribution

- In general, for a t-distribution, we compute the 95% confidence interval for the mean with the following formula:

$$95\%\ CI_{t\text{-}dis}=y_{mean}\pm(t_{95\%}\ u(y))$$

where

- y_{mean} is the sample mean
- $t_{95\%}$ is the number of standard errors (coverage factor), extending from the mean of a distribution required to contain 0.95 of the area
- u(y) is the estimated standard uncertainty.

- For large degrees of freedom, t-distribution becomes a Gaussian and 95% CI lies with in 1.96 SEs from the mean.

$$95\%\ CI_{gauss}=y_{mean}\pm(1.96*u(y))$$

Degrees of freedom, ν	Coverage factor, $t_{95\%}$
2	4.30
3	3.18
4	2.78
5	2.57
6	2.45
7	2.36
8	2.31
9	2.26
10	2.23
11	2.20
12	2.18
13	2.16
14	2.14
15	2.13
16	2.12
17	2.11
18	2.10
19	2.09
20	2.09
25	2.06
30	2.04
40	2.02
50	2.01
100	1.98
Infinite	1.96

The values of $t_{95\%}$ to be used in a confidence interval for different degrees of freedom ν, can be looked up in a table of the t distribution.

**95% Coverage factors, *$t_{95\%}$ as a function of the number*
of degrees of freedom, *ν***

Expanded uncertainty

- The measure of uncertainty incorporating the CI's is called **expanded uncertainty**, *U(y)*, and is obtained by multiplying the standard error, *u*(y) by a **coverage factor**, *t (obtained by using confidence intervals)*.

  $$U = tu(y)$$

- It is confidently believed that *y* is greater than or equal to $y - U$, and is less than or equal to $y + U$, which is commonly written as $Y = y \pm U$.

Welch–Satterthwaite equation

- *If* $y = f(x_1, x_2, \ldots, x_n)$*, the combined standard* uncertainty, $u(y)$*, in y as discussed earlier is*
- $u^2(y) = c_1^2 u^2(x_1) + c_2^2 u^2(x_2) + \cdots + c_n^2 u^2(x_n)$,

  where the *c's are sensitivity coefficients defined by the partial derivatives,*
- $c_i = \partial y / \partial x_i \; (i = 1, 2, \ldots, n)$.
- Each of the standard uncertainties, $u(x_i)$*, of the inputs,* x_i *, is associated with* v_i degrees of freedom. The obvious question now is as follows: how many degrees of freedom should we associate with *u(y) on the left-hand side of equation?*
- *The answer is* provided by the Welch–Satterthwaite formula which, though only approximate, is nevertheless adequate for most cases.

- The effective number of degrees of freedom, v_{eff} *for y is given by* Welch–Satterthwaite formula [2] (Section 10.3)as:

$$\nu_{\text{eff}} = \frac{u^4(y)}{\sum_{i=1}^{n} \frac{c_i^4 u^4(x_i)}{\nu_i}}.$$

- The effective number of degrees of freedom, v_{eff}, *is not necessarily an integer.* In practice, v_{eff} *is often truncated to an integer for the purpose of calculating a* coverage factor, $t_{95\%}$.

Evaluating uncertainty.. A brief overview

- Find the mean of the observations.
- Find the standard deviation, s and then the standard error $u(y_{mean})$in the mean.
- Find the best estimate of the measurand by adding and subtracting the systematic error, Z

  $\mathbf{y_{be} = y_{mean} \pm Z}$
- Find the combined standard uncertainty $\mathbf{u^2(y_{be}) = u^2(y_{mean}) + u^2(Z)}$ and calculate ν_{eff} .
- If y is a function of one or more variables, find the uncertainties in all those variables and calculate the combined uncertainty $u^2(y) = c_1^2 u^2(x_1) + c_2^2 u^2(x_2) + \cdots + c_n^2 u^2(x_n)$.
- *Calculate* ν_{eff} for combined uncertainties of all variables and sytematic errors.
- Find the coverage factor $t_{95\%}$, confidence interval and expanded uncertainty U(y) to find y±U(y).

Example 1

- The moment of inertia, *I , of a solid cylinder of mass M, rotating about its principal* axis, is given by, $I = MR^2$, where M_{mean}(n=8) =252.6g, $u(M_{mean})$=2.5g, R_{mean}(n=6)=6.35cm and u(R)=0.05cm.
- Use this information to determine

(a) the best estimate for the moment of inertia of the cylinder;

(b) the standard uncertainty in the best estimate of the moment of inertia

(c) the effective number of degrees of freedom of the measurand uncertainty using the Welch–Satterthwaite formula;

(d) the coverage factor for the 95% level of confidence; and

(e) the coverage interval containing *I at the 95% level of confidence.*

(a) the best estimate of the moment of inertia is

$$I = \frac{252.6 \times (6.35)^2}{2} = 5092.7 \text{ g} \cdot \text{cm}^2.$$

(b) the variance in the best estimate as

$$u^2(I) = c_M^2 u^2(M) + c_R^2 u^2(R),$$

where

$$c_M = \frac{\partial I}{\partial M} = \frac{R^2}{2}, \qquad c_R = \frac{\partial I}{\partial R} = MR.$$

$$\begin{aligned} u^2(I) &= \left(\frac{R^2}{2}\right)^2 u^2(M) + (MR)^2 u^2(R) \\ &= \left(\frac{(6.35)^2}{2}\right)^2 \times (2.5)^2 + (252.6 \times 6.35)^2 \times (0.05)^2 \\ &= 2540.5 + 6432.1 = 8972.6\,(\text{g} \cdot \text{cm}^2)^2. \end{aligned}$$

It follows that $u(I) = 94.7\text{g} \cdot \text{cm}^2$.

(c) the Welch–Satterthwaite formula for this example as

$$\nu_{\text{eff}} = \frac{\left[c_M^2 u^2(M) + c_R^2 u^2(R)\right]^2}{\dfrac{c_M^4 u^4(M)}{\nu_M} + \dfrac{c_R^4 u^4(R)}{\nu_R}},$$

where ν_M is the number of degrees of freedom in the calculation of the standard uncertainty in M, i.e. $\nu_M = 8 - 1 = 7$. The number of degrees of freedom ν_R in the calculation of the standard uncertainty in R is $\nu_R = 5 - 1 = 4$. From part (b), $c_M^2 u^2(M) = 2540.5$ and $c_R^2 u^2(R) = 6432.1$, so

$$\nu_{\text{eff}} = \frac{(2540.5 + 6432.1)^2}{\dfrac{(2540.5)^2}{7} + \dfrac{(6432.1)^2}{4}} = 7.1,$$

which truncates to 7 for the purpose of calculating the coverage factor, k.

(d) The t value for the 95% level of confidence and seven degrees of freedom is found from the table We have $k = t_{95\%,7} = 2.36$.

(e) The interval containing the true value at the 95% level of confidence is $5092.7\ \text{g} \cdot \text{cm}^2 \pm 2.36 \times 94.7\ \text{g} \cdot \text{cm}^2 = (5092.7 \pm 223.5)\ \text{g} \cdot \text{cm}^2$. The moment of inertia may be expressed in scientific notation to an appropriate number of significant figures as

$$\text{Moment of inertia of the cylinder} = (5.09 \pm 0.22) \times 10^3\,\text{g} \cdot \text{cm}^2.$$

Example 2

- The diameter of a wire is measured five times using a micrometer with a resolution of 0.01 mm. The mean diameter is found to be 0.253 mm with a standard uncertainty in the mean of 0.007 mm. Use this information to calculate

(a) the best estimate of the cross-sectional area of the wire;

(b) the standard uncertainty in the best estimate of the diameter;

(c) the effective number of degrees of freedom for the standard uncertainty in diameter;

(d) the coverage factor, $t_{95\%}$

(e) the coverage interval containing the true value of the cross-sectional area of the wire at the 95% level of confidence.

(a) The value of the cross-sectional area, A, of the wire is given by

$$A = \frac{\pi D^2}{4},$$

where D is diameter of the wire. D may be written as

$$D = X + Z.$$

X is the mean diameter of the wire obtained by calculating the mean of repeat values of the diameter. Z is the correction required due to systematic errors. From the information in this example, $X = 0.253$ mm. Since the correction term due to the resolution of the instrument is as likely to be positive as negative, we take $Z = 0$. It follows that $D = 0.253 \text{ mm} + 0 = 0.253 \text{ mm}$.

Substituting $D = 0.253$ mm gives $A = 0.0503 \text{ mm}^2$.

(b) The standard uncertainty in the diameter, $u(D)$, can be found using

$$u^2(D) = u^2(X) + u^2(Z).$$

$u(X)$ is given in the question as equal to 0.007 mm.

$u(Z_D) = 0.01 \text{ mm}/\sqrt{12} = 2.9 \times 10^{-3}$ mm.

$$\begin{aligned} u^2(D) &= (7 \times 10^{-3})^2 + (2.9 \times 10^{-3})^2 \\ &= 4.9 \times 10^{-5} + 8.33 \times 10^{-6} \\ &= 5.73 \times 10^{-5} \text{ mm}^2. \end{aligned}$$

It follows that $u(D) = 7.6 \times 10^{-3}$ mm.

(c) $$\nu_{\text{eff}} = \frac{\left[c_X^2 u^2(X) + c_Z^2 u^2(Z)\right]^2}{\frac{c_X^4 u^4(X)}{\nu_X} + \frac{c_Z^4 u^4(Z)}{\nu_Z}}.$$

$$c_X = \frac{\partial D}{\partial X} = 1$$

$$c_Z = \frac{\partial D}{\partial Z} = 1.$$

Therefore equation simplifies to

$$\nu_{\text{eff}} = \frac{[u^2(X) + u^2(Z)]^2}{\frac{u^4(X)}{\nu_X} + \frac{u^4(Z)}{\nu_Z}}.$$

Now $\nu_X = 5 - 1 = 4$. Since the uncertainty in the standard uncertainty in Z is zero, equation (10.14) indicates that the effective number of degrees of freedom is very large: ν_Z tends to ∞. Equation (10.36) becomes

$$\nu_{\text{eff}} = \frac{(5.73 \times 10^{-5})^2}{\frac{(7 \times 10^{-3})^4}{4} + 0} = 5.5,$$

which truncates to $\nu_{\text{eff}} = 5$.

(d) The t value for the 95% confidence interval for D based on five degrees of freedom is 2.57

(e) To calculate the coverage interval containing the true value of the area at the 95% level of confidence, we write

$$u(A) = \left(\frac{\partial A}{\partial D}\right) u(D).$$

$$\frac{\partial A}{\partial D} = \frac{\pi D}{2} = \frac{\pi \times 0.253}{2} = 0.397 \text{ mm},$$

so

$$u(A) = 0.397 \times 7.6 \times 10^{-3} = 0.0030 \text{ mm}^2.$$

It follows that the coverage interval containing the true value of the cross-sectional area at the 95% level of confidence is $(0.0503 \pm 2.571 \times 0.0030)$ mm^2, i.e.

$$\text{Cross-sectional area} = (0.0503 \pm 0.0077) \text{ mm}^2.$$

References

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2. Les Kirkup and Bob Frenkel, **An introduction to uncertainty in measurement**, Cambridge University press (2006).
3. http://onlinestatbook.com/chapter8/t_distribution.html
4. http://physics.nist.gov/cuu/Uncertainty/coverage.html