Chapter 3

Reflection and Refraction

As we know from everyday experience, when light arrives at an interface between materials it partially reflects and partially transmits. In this chapter, we examine what happens to plane waves when they propagate from one material (characterized by indices n or even by complex index $\mathcal{N}$) to another material. We will derive expressions to quantify the amount of reflection and transmission. The results depend on the angle of incidence (i.e. the angle between $\mathbf{k}$ and the surface normal) as well as on the orientation of the electric field (called polarization – not to be confused with $\mathbf{P}$, also called polarization). In this chapter, we consider only isotropic materials (e.g. glass); in chapter 5 we consider anisotropic materials (e.g. a crystal).

As we develop the connection between incident, reflected, and transmitted light waves,[1] several familiar relationships will emerge naturally (e.g. Snell's law and Brewster's angle). The formalism also describes polarization-dependent phase shifts upon reflection (especially interesting in the case of reflections from metals).

For simplicity, we initially neglect the imaginary part of the refractive index. Each plane wave is thus characterized by a real wave vector $\mathbf{k}$. We will write each plane wave in the form $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 \exp\left[i\left(\mathbf{k}\cdot\mathbf{r} - \omega t\right)\right]$, where, as usual, only the real part of the field corresponds to the physical field. The restriction to real refractive indices is not as serious as it might seem. The use of the letter n instead of $\mathcal{N}$ hardly matters. The math is all the same, which demonstrates the power of the complex notation. We can simply update our expressions in the end to include complex refractive indices, but in the mean time it is easier to think of absorption as negligible.

3.1 Refraction at an Interface

Consider a planar boundary between two materials with different indices. Let index n_i characterize the material on the left, and the index n_t characterize the

[1] See M. Born and E. Wolf, *Principles of Optics*, 7th ed., Sect. 1.5 (Cambridge University Press, 1999).

material on the right, as depicted in the Fig. 3.1. When a plane wave traveling in the direction $\mathbf{k}_i$ is incident on the boundary from the left, it gives rise to a reflected plane wave traveling in the direction $\mathbf{k}_r$ and a transmitted plane wave traveling in the direction $\mathbf{k}_t$. The incident and reflected waves exist only to the left of the material interface, and the transmitted wave exists only to the right of the interface. The angles θ_i, θ_r, and θ_t give the angles that each respective wave vector ($\mathbf{k}_i$, $\mathbf{k}_r$, and $\mathbf{k}_t$) makes with the normal to the interface.

For simplicity, we'll assume that both of the materials are isotropic here. (Chapter 5 discusses refraction for anisotropic materials.) In this case, $\mathbf{k}_i$, $\mathbf{k}_r$, and $\mathbf{k}_t$ all lie in a single plane, referred to as the *plane of incidence*, (i.e. the plane represented by the surface of this page). We are free to orient our coordinate system in many different ways (and every textbook seems to do it differently!).[2] We choose the y–z plane to be the plane of incidence, with the z-direction normal to the interface and the x-axis pointing into the page.

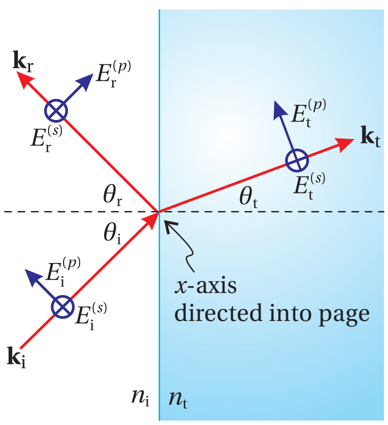

Figure 3.1 Incident, reflected, and transmitted plane wave fields at a material interface.

The electric field vector for each plane wave is confined to a plane perpendicular to its wave vector. We are free to decompose the field vector into arbitrary components as long as they are perpendicular to the wave vector. It is customary to choose one of the electric field vector components to be that which lies within the plane of incidence. We call this *p-polarized light*, where p stands for *parallel* to the plane of incidence. The remaining electric field vector component is directed normal to the plane of incidence and is called *s-polarized light*. The s stands for *senkrecht*, a German word meaning perpendicular.

Using this system, we can decompose the electric field vector $\mathbf{E}_i$ into its p-polarized component $E_i^{(p)}$ and its s-polarized component $E_i^{(s)}$, as depicted in Fig. 3.1. The s component $E_i^{(s)}$ is represented by the tail of an arrow pointing into the page, or the x-direction in our convention. The other fields $\mathbf{E}_r$ and $\mathbf{E}_t$ are similarly split into s and p components as indicated in Fig. 3.1. All field components are considered to be positive when they point in the direction of their respective arrows.[3] Note that the s-polarized components are parallel for all three plane waves, whereas the p-polarized components are not (except at normal incidence) because each plane wave travels in a different direction.

By inspection of Fig. 3.1, we can write the various wave vectors in terms of the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ unit vectors:

$$\begin{aligned}\mathbf{k}_i &= k_i\left(\hat{\mathbf{y}}\sin\theta_i + \hat{\mathbf{z}}\cos\theta_i\right)\\ \mathbf{k}_r &= k_r\left(\hat{\mathbf{y}}\sin\theta_r - \hat{\mathbf{z}}\cos\theta_r\right)\\ \mathbf{k}_t &= k_t\left(\hat{\mathbf{y}}\sin\theta_t + \hat{\mathbf{z}}\cos\theta_t\right)\end{aligned} \tag{3.1}$$

Also by inspection of Fig. 3.1 (following the conventions for the electric fields indicated by the arrows), we can write the incident, reflected, and transmitted

[2]For example, our convention is different than that used by E. Hecht, *Optics*, 3rd ed., Sect. 4.6.2 (Massachusetts: Addison-Wesley, 1998).

[3]Many textbooks draw the arrow for $E_r^{(p)}$ in the direction opposite of ours. However, that choice leads to an awkward situation at normal incidence (i.e. $\theta_i = \theta_r = 0$) where the arrows for the incident and reflected fields are parallel for the s-component but anti parallel for the p-component.

fields in terms of $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$:

$$\begin{aligned}
\mathbf{E}_{\mathrm{i}} &= \left[E_{\mathrm{i}}^{(p)}\left(\hat{\mathbf{y}}\cos\theta_{\mathrm{i}} - \hat{\mathbf{z}}\sin\theta_{\mathrm{i}}\right) + \hat{\mathbf{x}}E_{\mathrm{i}}^{(s)}\right] e^{i\left[k_{\mathrm{i}}(y\sin\theta_{\mathrm{i}} + z\cos\theta_{\mathrm{i}}) - \omega_{\mathrm{i}} t\right]} \\
\mathbf{E}_{\mathrm{r}} &= \left[E_{\mathrm{r}}^{(p)}\left(\hat{\mathbf{y}}\cos\theta_{\mathrm{r}} + \hat{\mathbf{z}}\sin\theta_{\mathrm{r}}\right) + \hat{\mathbf{x}}E_{\mathrm{r}}^{(s)}\right] e^{i\left[k_{\mathrm{r}}(y\sin\theta_{\mathrm{r}} - z\cos\theta_{\mathrm{r}}) - \omega_{\mathrm{r}} t\right]} \\
\mathbf{E}_{\mathrm{t}} &= \left[E_{\mathrm{t}}^{(p)}\left(\hat{\mathbf{y}}\cos\theta_{\mathrm{t}} - \hat{\mathbf{z}}\sin\theta_{\mathrm{t}}\right) + \hat{\mathbf{x}}E_{\mathrm{t}}^{(s)}\right] e^{i\left[k_{\mathrm{t}}(y\sin\theta_{\mathrm{t}} + z\cos\theta_{\mathrm{t}}) - \omega_{\mathrm{t}} t\right]}
\end{aligned} \tag{3.2}$$

Each field has the form (2.8). We have utilized the k-vectors (3.1) in the exponents of (3.2).

Now we are ready to connect the fields on one side of the interface to the fields on the other side. This is done using *boundary conditions*. As explained in appendix 3.A, Maxwell's equations require the components of $\mathbf{E}$ that are parallel to the interface to be the same on either side of the boundary. In our coordinate system, the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components are parallel to the interface, whereas $z = 0$ defines the interface. This means that at $z = 0$ the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components of the combined incident and reflected fields must equal the corresponding components of the transmitted field:

$$\begin{aligned}
\left[E_{\mathrm{i}}^{(p)}\hat{\mathbf{y}}\cos\theta_{\mathrm{i}} + \hat{\mathbf{x}}E_{\mathrm{i}}^{(s)}\right] e^{i(k_{\mathrm{i}} y\sin\theta_{\mathrm{i}} - \omega_{\mathrm{i}} t)} &+ \left[E_{\mathrm{r}}^{(p)}\hat{\mathbf{y}}\cos\theta_{\mathrm{r}} + \hat{\mathbf{x}}E_{\mathrm{r}}^{(s)}\right] e^{i(k_{\mathrm{r}} y\sin\theta_{\mathrm{r}} - \omega_{\mathrm{r}} t)} \\
&= \left[E_{\mathrm{t}}^{(p)}\hat{\mathbf{y}}\cos\theta_{\mathrm{t}} + \hat{\mathbf{x}}E_{\mathrm{t}}^{(s)}\right] e^{i(k_{\mathrm{t}} y\sin\theta_{\mathrm{t}} - \omega_{\mathrm{t}} t)}
\end{aligned} \tag{3.3}$$

Since this equation must hold for all conceivable values of t and y, we are compelled to set all the phase factors in the complex exponentials equal to each other. The time portion of the phase factors requires the frequency of all waves to be the same:

$$\omega_{\mathrm{i}} = \omega_{\mathrm{r}} = \omega_{\mathrm{t}} \equiv \omega \tag{3.4}$$

(We could have guessed that all frequencies would be the same; otherwise wave fronts would be annihilated or created at the interface.) Similarly, equating the spatial terms in the exponents of (3.3) requires

$$k_{\mathrm{i}}\sin\theta_{\mathrm{i}} = k_{\mathrm{r}}\sin\theta_{\mathrm{r}} = k_{\mathrm{t}}\sin\theta_{\mathrm{t}} \tag{3.5}$$

Now recall from (2.19) the relations $k_{\mathrm{i}} = k_{\mathrm{r}} = n_{\mathrm{i}}\omega/c$ and $k_{\mathrm{t}} = n_{\mathrm{t}}\omega/c$. With these relations, (3.5) yields the *law of reflection*

$$\theta_{\mathrm{r}} = \theta_{\mathrm{i}} \tag{3.6}$$

and *Snell's law*

$$n_{\mathrm{i}}\sin\theta_{\mathrm{i}} = n_{\mathrm{t}}\sin\theta_{\mathrm{t}} \tag{3.7}$$

The three angles θ_{i}, θ_{r}, and θ_{t} are not independent. The reflected angle matches the incident angle, and the transmitted angle obeys Snell's law. The phenomenon of *refraction* refers to the fact that θ_{i} and θ_{t} are different. That is, light 'bends' as it transmits through an interface.

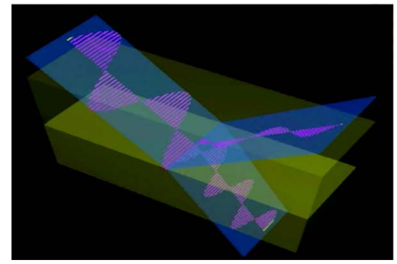

Figure 3.2 Animation of s- and p-polarized fields incident on an interface as the angle of incidence is varied.

Willebrord Snell (or Snellius) (1580–1626, Dutch) was an astronomer and mathematician born in Leiden, Netherlands. In 1613 he succeeded his father as professor of mathematics at the University of Leiden. He was an accomplished mathematician, developing a new method for calculating π as well as an improved method for measuring the circumference of the earth. He is most famous for his rediscovery of the law of refraction in 1621. (The law was known (in table form) to the ancient Greek mathematician Ptolemy, to Persian engineer Ibn Sahl (900s), and to Polish philosopher Witelo (1200s).) Snell authored several books, including one on trigonometry, published a year after his death. (Wikipedia)

Because the exponents are all identical, (3.3) reduces to two relatively simple equations (one for each dimension, $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$):

$$E_{\mathrm{i}}^{(s)} + E_{\mathrm{r}}^{(s)} = E_{\mathrm{t}}^{(s)} \tag{3.8}$$

and

$$\left(E_{\mathrm{i}}^{(p)} + E_{\mathrm{r}}^{(p)}\right)\cos\theta_{\mathrm{i}} = E_{\mathrm{t}}^{(p)}\cos\theta_{\mathrm{t}} \tag{3.9}$$

We have derived these equations from the boundary condition (3.54) on the parallel component of the electric field. This set of equations has four unknowns ($E_{\mathrm{r}}^{(p)}$, $E_{\mathrm{r}}^{(s)}$, $E_{\mathrm{t}}^{(p)}$, and $E_{\mathrm{t}}^{(s)}$), assuming that we pick the incident fields. We require two additional equations to solve the system. These are obtained using the separate boundary condition on the parallel component of magnetic fields given in (3.58) (also discussed in appendix 3.A).

From Faraday's law (1.3), we have for a plane wave (see (2.56))

$$\mathbf{B} = \frac{\mathbf{k}\times\mathbf{E}}{\omega} = \frac{n}{c}\hat{\mathbf{u}}\times\mathbf{E} \tag{3.10}$$

where $\hat{\mathbf{u}} \equiv \mathbf{k}/k$ is a unit vector in the direction of $\mathbf{k}$. We have also utilized (2.19) for a real index. This expression is useful for writing $\mathbf{B}_{\mathrm{i}}$, $\mathbf{B}_{\mathrm{r}}$, and $\mathbf{B}_{\mathrm{t}}$ in terms of the electric field components that we have already introduced. When injecting (3.1) and (3.2) into (3.10), the incident, reflected, and transmitted magnetic fields turn out to be

$$\begin{aligned}
\mathbf{B}_{\mathrm{i}} &= \frac{n_{\mathrm{i}}}{c}\left[-\hat{\mathbf{x}}E_{\mathrm{i}}^{(p)} + E_{\mathrm{i}}^{(s)}\left(-\hat{\mathbf{z}}\sin\theta_{\mathrm{i}} + \hat{\mathbf{y}}\cos\theta_{\mathrm{i}}\right)\right]e^{i\left[k_{\mathrm{i}}\left(y\sin\theta_{\mathrm{i}}+z\cos\theta_{\mathrm{i}}\right)-\omega_{\mathrm{i}}t\right]}\\
\mathbf{B}_{\mathrm{r}} &= \frac{n_{\mathrm{r}}}{c}\left[\hat{\mathbf{x}}E_{\mathrm{r}}^{(p)} + E_{\mathrm{r}}^{(s)}\left(-\hat{\mathbf{z}}\sin\theta_{\mathrm{r}} - \hat{\mathbf{y}}\cos\theta_{\mathrm{r}}\right)\right]e^{i\left[k_{\mathrm{r}}\left(y\sin\theta_{\mathrm{r}}-z\cos\theta_{\mathrm{r}}\right)-\omega_{\mathrm{r}}t\right]}\\
\mathbf{B}_{\mathrm{t}} &= \frac{n_{\mathrm{t}}}{c}\left[-\hat{\mathbf{x}}E_{\mathrm{t}}^{(p)} + E_{\mathrm{t}}^{(s)}\left(-\hat{\mathbf{z}}\sin\theta_{\mathrm{t}} + \hat{\mathbf{y}}\cos\theta_{\mathrm{t}}\right)\right]e^{i\left[k_{\mathrm{t}}\left(y\sin\theta_{\mathrm{t}}+z\cos\theta_{\mathrm{t}}\right)-\omega_{\mathrm{t}}t\right]}
\end{aligned} \tag{3.11}$$

Next, we apply the boundary condition (3.58), namely that the components of $\mathbf{B}$ parallel to the interface (i.e. in the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ dimensions) are the same[4] on either side of the plane $z = 0$. Since we already know that the exponents are all equal and that $\theta_{\mathrm{r}} = \theta_{\mathrm{i}}$ and $n_{\mathrm{i}} = n_{\mathrm{r}}$, the boundary condition gives

$$\frac{n_{\mathrm{i}}}{c}\left[-\hat{\mathbf{x}}E_{\mathrm{i}}^{(p)} + E_{\mathrm{i}}^{(s)}\hat{\mathbf{y}}\cos\theta_{\mathrm{i}}\right] + \frac{n_{\mathrm{i}}}{c}\left[\hat{\mathbf{x}}E_{\mathrm{r}}^{(p)} - E_{\mathrm{r}}^{(s)}\hat{\mathbf{y}}\cos\theta_{\mathrm{i}}\right] = \frac{n_{\mathrm{t}}}{c}\left[-\hat{\mathbf{x}}E_{\mathrm{t}}^{(p)} + E_{\mathrm{t}}^{(s)}\hat{\mathbf{y}}\cos\theta_{\mathrm{t}}\right] \tag{3.12}$$

As before, (3.12) reduces to two relatively simple equations (one for the $\hat{\mathbf{x}}$ dimension and one for the $\hat{\mathbf{y}}$ dimension):

$$n_{\mathrm{i}}\left(E_{\mathrm{i}}^{(p)} - E_{\mathrm{r}}^{(p)}\right) = n_{\mathrm{t}}E_{\mathrm{t}}^{(p)} \tag{3.13}$$

and

$$n_{\mathrm{i}}\left(E_{\mathrm{i}}^{(s)} - E_{\mathrm{r}}^{(s)}\right)\cos\theta_{\mathrm{i}} = n_{\mathrm{t}}E_{\mathrm{t}}^{(s)}\cos\theta_{\mathrm{t}} \tag{3.14}$$

These two equations together with (3.8) and (3.9) allow us to solve for the reflected $\mathbf{E}_{\mathrm{r}}$ and transmitted fields $\mathbf{E}_{\mathrm{t}}$ for the s and p polarization components. However, (3.8), (3.9), (3.13), and (3.14) are not yet in their most convenient form.

[4]We assume the permeability μ_0 is the same everywhere—no magnetic effects.

3.2 The Fresnel Coefficients

Augustin Fresnel first derived the equations in the previous section. Since he lived well before Maxwell's time, he did not have the benefit of Maxwell's equations as we have. Instead, Fresnel thought of light as transverse mechanical waves propagating within materials. (Fresnel was naturally a proponent of luminiferous ether.) Instead of relating the parallel components of the electric and magnetic fields across the boundary between the materials, Fresnel used the principle that the two materials should not slip relative to each other at the boundary. This 'gluing' of the materials at the interface also forbids the possibility of gaps or the like forming at the interface as the two materials experience wave vibrations. This mechanical approach to light worked splendidly, arriving at the same results that we obtained from our modern viewpoint.

Fresnel wrote the relationships between the various plane waves depicted in Fig. 3.1 in terms of coefficients that compare the reflected and transmitted field amplitudes to those of the incident field. In the following example, we illustrate this procedure for s-polarized light. It is left as a homework exercise to solve the equations for p-polarized light (see P3.1).

Augustin Fresnel (1788–1829, French) was born in Broglie, France, the son of an architect. As a child, he was slow to develop and still could not read when he was eight years old, but by age sixteen he excelled and entered the École Polytechnique where he earned distinction. As a young man, Fresnel began a successful career as an engineer, but he lost his post in 1814 when Napoleon returned to power. (Fresnel had supported the Bourbons.) This difficult year was when Fresnel turned his attention to optics. Fresnel became a major proponent of the wave theory of light and four years later wrote a paper on diffraction for which he was awarded a prize by the French Academy of Sciences. A year later he was appointed commissioner of lighthouses, which motivated the invention of the Fresnel lens (still used in many commercial applications). Fresnel was under appreciated before his untimely death from tuberculosis. Many of his papers did not make it into print until years later. Fresnel made huge advances in the understanding of reflection, diffraction, polarization, and birefringence. In 1824 Fresnel wrote to Thomas Young, "All the compliments that I have received from Arago, Laplace and Biot never gave me so much pleasure as the discovery of a theoretic truth, or the confirmation of a calculation by experiment." Augustin Fresnel is a hero of one of the authors of this textbook. (Wikipedia)

Example 3.1

Calculate the ratio of transmitted field to the incident field and the ratio of the reflected field to incident field for s-polarized light.

Solution: We write (3.8) and (3.14) as

$$E_i^{(s)} + E_r^{(s)} = E_t^{(s)} \quad \text{and} \quad E_i^{(s)} - E_r^{(s)} = \frac{n_t \cos\theta_t}{n_i \cos\theta_i} E_t^{(s)} \tag{3.15}$$

Adding these two equations yields

$$2E_i^{(s)} = \left[1 + \frac{n_t \cos\theta_t}{n_i \cos\theta_i}\right] E_t^{(s)} \tag{3.16}$$

After a little rearrangement we get

$$\frac{E_t^{(s)}}{E_i^{(s)}} = \frac{2n_i \cos\theta_i}{n_i \cos\theta_i + n_t \cos\theta_t} \tag{3.17}$$

To get the ratio of reflected field to incident field, we subtract the equations in (3.15) to get

$$2E_r^{(s)} = \left[1 - \frac{n_t \cos\theta_t}{n_i \cos\theta_i}\right] E_t^{(s)} \tag{3.18}$$

We divide (3.18) by (3.16), and after simplification arrive at

$$\frac{E_r^{(s)}}{E_i^{(s)}} = \frac{n_i \cos\theta_i - n_t \cos\theta_t}{n_i \cos\theta_i + n_t \cos\theta_t} \tag{3.19}$$

The ratio of the reflected and transmitted field components to the incident field components are specified by the *Fresnel coefficients*, which are defined as follows:

$$r_s \equiv \frac{E_{\mathrm{r}}^{(s)}}{E_{\mathrm{i}}^{(s)}} = \frac{n_{\mathrm{i}}\cos\theta_{\mathrm{i}} - n_{\mathrm{t}}\cos\theta_{\mathrm{t}}}{n_{\mathrm{i}}\cos\theta_{\mathrm{i}} + n_{\mathrm{t}}\cos\theta_{\mathrm{t}}} = \frac{\sin\theta_{\mathrm{t}}\cos\theta_{\mathrm{i}} - \sin\theta_{\mathrm{i}}\cos\theta_{\mathrm{t}}}{\sin\theta_{\mathrm{t}}\cos\theta_{\mathrm{i}} + \sin\theta_{\mathrm{i}}\cos\theta_{\mathrm{t}}} = \frac{\sin(\theta_{\mathrm{t}} - \theta_{\mathrm{i}})}{\sin(\theta_{\mathrm{t}} + \theta_{\mathrm{i}})} \tag{3.20}$$

$$t_s \equiv \frac{E_{\mathrm{t}}^{(s)}}{E_{\mathrm{i}}^{(s)}} = \frac{2n_{\mathrm{i}}\cos\theta_{\mathrm{i}}}{n_{\mathrm{i}}\cos\theta_{\mathrm{i}} + n_{\mathrm{t}}\cos\theta_{\mathrm{t}}} = \frac{2\sin\theta_{\mathrm{t}}\cos\theta_{\mathrm{i}}}{\sin\theta_{\mathrm{t}}\cos\theta_{\mathrm{i}} + \sin\theta_{\mathrm{i}}\cos\theta_{\mathrm{t}}} = \frac{2\sin\theta_{\mathrm{t}}\cos\theta_{\mathrm{i}}}{\sin(\theta_{\mathrm{t}} + \theta_{\mathrm{i}})} \tag{3.21}$$

$$r_p \equiv \frac{E_{\mathrm{r}}^{(p)}}{E_{\mathrm{i}}^{(p)}} = \frac{n_{\mathrm{i}}\cos\theta_{\mathrm{t}} - n_{\mathrm{t}}\cos\theta_{\mathrm{i}}}{n_{\mathrm{i}}\cos\theta_{\mathrm{t}} + n_{\mathrm{t}}\cos\theta_{\mathrm{i}}} = \frac{\sin\theta_{\mathrm{t}}\cos\theta_{\mathrm{t}} - \sin\theta_{\mathrm{i}}\cos\theta_{\mathrm{i}}}{\sin\theta_{\mathrm{t}}\cos\theta_{\mathrm{t}} + \sin\theta_{\mathrm{i}}\cos\theta_{\mathrm{i}}} = \frac{\tan(\theta_{\mathrm{t}} - \theta_{\mathrm{i}})}{\tan(\theta_{\mathrm{t}} + \theta_{\mathrm{i}})} \tag{3.22}$$

$$t_p \equiv \frac{E_{\mathrm{t}}^{(p)}}{E_{\mathrm{i}}^{(p)}} = \frac{2n_{\mathrm{i}}\cos\theta_{\mathrm{i}}}{n_{\mathrm{i}}\cos\theta_{\mathrm{t}} + n_{\mathrm{t}}\cos\theta_{\mathrm{i}}} = \frac{2\sin\theta_{\mathrm{t}}\cos\theta_{\mathrm{i}}}{\sin\theta_{\mathrm{t}}\cos\theta_{\mathrm{t}} + \sin\theta_{\mathrm{i}}\cos\theta_{\mathrm{i}}} = \frac{2\sin\theta_{\mathrm{t}}\cos\theta_{\mathrm{i}}}{\sin(\theta_{\mathrm{t}} + \theta_{\mathrm{i}})\cos(\theta_{\mathrm{t}} - \theta_{\mathrm{i}})} \tag{3.23}$$

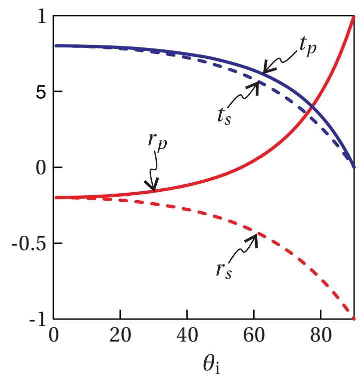

Figure 3.3 The Fresnel coefficients plotted versus θ_{i} for the case of an air-glass interface with $n_{\mathrm{i}} = 1$ and $n_{\mathrm{t}} = 1.5$.

All of the above forms of the Fresnel coefficients are potentially useful, depending on the problem at hand. Remember that the angles in the coefficient are not independently chosen, but are subject to Snell's law (3.7). Snell's law has been used to produce the alternative expressions from the first.

The Fresnel coefficients pin down the electric field amplitudes on the two sides of the boundary. They also keep track of phase shifts at a boundary. In Fig. 3.3 we have plotted the Fresnel coefficients for the case of an air-glass interface. Notice that the reflection coefficients are sometimes negative in this plot, which corresponds to a phase shift of π upon reflection (note $e^{i\pi} = -1$). Later we will see that when absorbing materials are encountered, more complicated phase shifts can arise due to the complex index of refraction.

3.3 Reflectance and Transmittance

We often want to know the fraction of power that reflects from or transmits through an interface. Energy conservation requires the incident power to balance the reflected and transmitted power:

$$P_{\mathrm{i}} = P_{\mathrm{r}} + P_{\mathrm{t}} \tag{3.24}$$

Moreover, the power separates cleanly into power associated with s- and p-polarized fields:

$$P_{\mathrm{i}}^{(\mathrm{s})} = P_{\mathrm{r}}^{(\mathrm{s})} + P_{\mathrm{t}}^{(\mathrm{s})} \quad \text{and} \quad P_{\mathrm{i}}^{(\mathrm{p})} = P_{\mathrm{r}}^{(\mathrm{p})} + P_{\mathrm{t}}^{(\mathrm{p})} \tag{3.25}$$

Since power is proportional to intensity (i.e. power per area) and intensity is proportional to the square of the field amplitude. We can write the fraction of reflected power, called *reflectance*, in terms of our previously defined Fresnel coefficients:

$$R_s \equiv \frac{P_\mathrm{r}^{(s)}}{P_\mathrm{i}^{(s)}} = \frac{I_\mathrm{r}^{(s)}}{I_\mathrm{i}^{(s)}} = \frac{\left|E_\mathrm{r}^{(s)}\right|^2}{\left|E_\mathrm{i}^{(s)}\right|^2} = |r_s|^2 \qquad \text{and} \qquad R_p \equiv \frac{P_\mathrm{r}^{(p)}}{P_\mathrm{i}^{(p)}} = \frac{I_\mathrm{r}^{(p)}}{I_\mathrm{i}^{(p)}} = \frac{\left|E_\mathrm{r}^{(p)}\right|^2}{\left|E_\mathrm{i}^{(p)}\right|^2} = \left|r_p\right|^2 \tag{3.26}$$

The total reflected intensity is therefore

$$I_\mathrm{r} = I_\mathrm{r}^{(s)} + I_\mathrm{r}^{(p)} = R_s I_\mathrm{i}^{(s)} + R_p I_\mathrm{i}^{(p)} \tag{3.27}$$

where, according to (2.62), the total incident intensity is given by

$$I_\mathrm{i} = I_\mathrm{i}^{(s)} + I_\mathrm{i}^{(p)} = \frac{1}{2} n_\mathrm{i} \epsilon_0 c \left[\left|E_\mathrm{i}^{(s)}\right|^2 + \left|E_\mathrm{i}^{(p)}\right|^2\right] \tag{3.28}$$

From (3.25) and (3.26), the transmitted power is

$$P_\mathrm{t}^{(\mathrm{s})} = P_\mathrm{i}^{(\mathrm{s})} - P_\mathrm{r}^{(\mathrm{s})} = (1 - R_s) P_\mathrm{i}^{(\mathrm{s})} \qquad \text{and} \qquad P_\mathrm{t}^{(\mathrm{p})} = P_\mathrm{i}^{(\mathrm{p})} + P_\mathrm{r}^{(\mathrm{p})} = \left(1 - R_p\right) P_\mathrm{i}^{(\mathrm{p})} \tag{3.29}$$

From this expression we see that the fraction of the power that transmits, called the *transmittance*, is

$$T_s \equiv \frac{P_\mathrm{t}^{(s)}}{P_\mathrm{i}^{(s)}} = 1 - R_s \qquad \text{and} \qquad T_p \equiv \frac{P_\mathrm{t}^{(p)}}{P_\mathrm{i}^{(p)}} = 1 - R_p \tag{3.30}$$

Figure 3.4 shows typical reflectance and transmittance values for an air-glass interface.

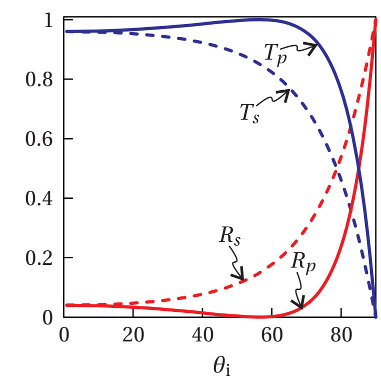

Figure 3.4 The reflectance and transmittance plotted versus θ_i for the case of an air-glass interface with $n_\mathrm{i} = 1$ and $n_\mathrm{t} = 1.5$.

You might be surprised at first to learn that

$$T_s \neq |t_s|^2 \qquad \text{and} \qquad T_p \neq \left|t_p\right|^2 \tag{3.31}$$

However, recall that the transmitted intensity (in terms of the transmitted fields) depends also on the refractive index. The Fresnel coefficients t_s and t_p relate the bare electric fields to each other, whereas the transmitted intensity is

$$I_\mathrm{t} = I_\mathrm{t}^{(s)} + I_\mathrm{t}^{(p)} = \frac{1}{2} n_\mathrm{t} \epsilon_0 c \left[\left|E_\mathrm{t}^{(s)}\right|^2 + \left|E_\mathrm{t}^{(p)}\right|^2\right] \tag{3.32}$$

In view of (3.28) and (3.32), we expect T_s and T_p to depend on the ratio of the refractive indices n_t and n_i in addition to $|t_s|^2$ or $\left|t_p\right|^2$.

There is another more subtle reason for the inequalities in (3.31). Consider a lateral strip of light associated with a plane wave incident upon the material interface in Fig. 3.5. Upon refraction into the second medium, the strip is seen to change its width by the factor $\cos\theta_\mathrm{t}/\cos\theta_\mathrm{i}$. This is a purely geometrical effect, owing to the change in propagation direction at the interface. Since power is intensity times area, the transmittance picks up this geometrical factor via the ratio of the areas $A_\mathrm{t}/A_\mathrm{i}$ as follows:

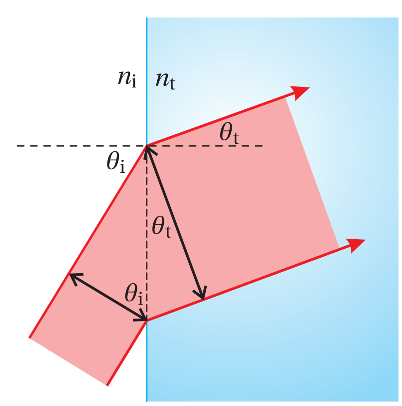

Figure 3.5 Light refracting into a surface

$$\begin{aligned} T_s &\equiv \frac{P_\mathrm{r}^{(s)}}{P_\mathrm{i}^{(s)}} = \frac{I_\mathrm{r}^{(s)} A_\mathrm{t}}{I_\mathrm{i}^{(s)} A_\mathrm{i}} = \frac{n_\mathrm{t} \cos\theta_\mathrm{t}}{n_\mathrm{i} \cos\theta_\mathrm{i}} |t_s|^2 \\ T_p &\equiv \frac{P_\mathrm{r}^{(p)}}{P_\mathrm{i}^{(p)}} = \frac{I_\mathrm{r}^{(p)} A_\mathrm{t}}{I_\mathrm{i}^{(p)} A_\mathrm{i}} = \frac{n_\mathrm{t} \cos\theta_\mathrm{t}}{n_\mathrm{i} \cos\theta_\mathrm{i}} \left|t_p\right|^2 \end{aligned} \qquad \text{(not valid if total internal reflection)} \tag{3.33}$$

David Brewster (1781–1868, Scottish) was born in Jedburgh, Scotland. His father was a teacher and wanted David to become a clergyman. At age twelve, David went to the University of Edinburgh for that purpose, but his inclination for natural science soon became apparent. He became licensed to preach, but his interests in science distracted him from that profession, and he spent much of his time studying diffraction. Taking an empirical approach, Brewster independently discovered many of the same things usually credited to Fresnel. He even made a dioptric apparatus for lighthouses before Fresnel developed his. Brewster became somewhat famous in his day for the development of the kaleidoscope and stereoscope for enjoyment by the general public. Brewster was a prolific science writer and editor throughout his life. Among his works is an important biography of Isaac Newton. He was knighted for his accomplishments in 1831. (Wikipedia)

Note that (3.33) is valid only if a real angle θ_t exists; it does not hold when the incident angle exceeds the critical angle for total internal reflection, discussed in section 3.5. In that situation, we must stick with (3.30).

Example 3.2

Show analytically that $R_p + T_p = 1$, where R_p is given by (3.26) and T_p is given by (3.33).

Solution: From (3.22) we have

$$R_p = \left|\frac{n_i\cos\theta_t - n_t\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i}\right|^2 = \frac{n_i^2\cos^2\theta_t - 2n_i n_t\cos\theta_i\cos\theta_t + n_i^2\cos^2\theta_i}{(n_i\cos\theta_t + n_t\cos\theta_i)^2}$$

From (3.23) and (3.33) we have

$$T_p = \frac{n_t\cos\theta_t}{n_i\cos\theta_i}\left|\frac{2n_i\cos\theta_i}{n_i\cos\theta_t + n_t\cos\theta_i}\right|^2 = \frac{4n_i n_t\cos\theta_i\cos\theta_t}{(n_i\cos\theta_t + n_t\cos\theta_i)^2}$$

Then

$$R_p + T_p = \frac{n_i^2\cos^2\theta_t + 2n_i n_t\cos\theta_i\cos\theta_t + n_i^2\cos^2\theta_i}{(n_i\cos\theta_t + n_t\cos\theta_i)^2} = \frac{(n_i\cos\theta_t + n_t\cos\theta_i)^2}{(n_i\cos\theta_t + n_t\cos\theta_i)^2} = 1$$

3.4 Brewster's Angle

Notice r_p and R_p go to zero at a certain angle in Figs. 3.3 and 3.4, indicating that no p-polarized light is reflected at this angle. This behavior is quite general, as we can see from the final form of the Fresnel coefficient formula for r_p in (3.22), which has $\tan(\theta_i + \theta_t)$ in the denominator. Since the tangent 'blows up' at $\pi/2$, the reflection coefficient goes to zero when

$$\theta_i + \theta_t = \frac{\pi}{2} \qquad \text{(requirement for zero } p\text{-polarized reflection)} \tag{3.34}$$

By inspecting Fig. 3.1, we see that this condition occurs when the reflected and transmitted wave vectors, $\mathbf{k}_r$ and $\mathbf{k}_t$, are perpendicular to each other (see also Fig. 3.6). If we insert (3.34) into Snell's law (3.7), we can solve for the incident angle θ_i that gives rise to this special circumstance:

$$n_i\sin\theta_i = n_t\sin\left(\frac{\pi}{2} - \theta_i\right) = n_t\cos\theta_i \tag{3.35}$$

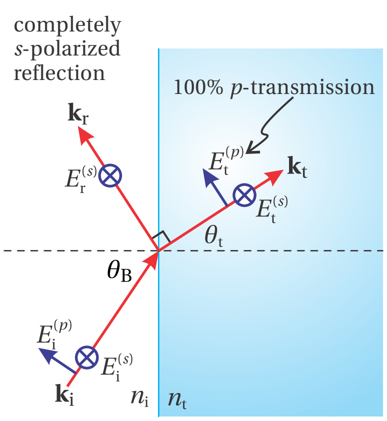

Figure 3.6 Brewster's angle coincides with the situation where $\mathbf{k}_r$ and $\mathbf{k}_t$ are perpendicular.

The angle that satisfies this equation, in terms of the refractive indices, is readily found to be

$$\theta_B = \tan^{-1}\frac{n_t}{n_i} \tag{3.36}$$

We have replaced the specific θ_i with θ_B in honor of Sir David Brewster who first discovered the phenomenon. The angle θ_B is called *Brewster's angle*. At Brewster's angle, no p-polarized light reflects (see L 3.4). Physically, the p-polarized light cannot reflect because $\mathbf{k}_r$ and $\mathbf{k}_t$ are perpendicular. A reflection would require the microscopic dipoles at the surface of the second material to radiate along their axes, which they cannot do. Maxwell's equations 'know' about this, and so everything is nicely consistent.

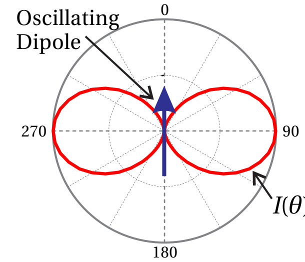

Figure 3.7 The intensity radiation pattern of an oscillating dipole as a function of angle. Note that the dipole does not radiate along the axis of oscillation, giving rise to Brewster's angle for reflection.

3.5 Total Internal Reflection

From Snell's law (3.7), we can compute the transmitted angle in terms of the incident angle:

$$\theta_t = \sin^{-1}\left(\frac{n_i}{n_t}\sin\theta_i\right) \tag{3.37}$$

The angle θ_t is real only if the argument of the inverse sine is less than or equal to one. If $n_i > n_t$, we can find a *critical angle* beyond which the argument begins to exceed one:

$$\theta_c \equiv \sin^{-1}\frac{n_t}{n_i} \tag{3.38}$$

When $\theta_i > \theta_c$, then there is *total internal reflection* and we can directly show that $R_s = 1$ and $R_p = 1$ (see P3.9).[5] To demonstrate this, one computes the Fresnel coefficients (3.20) and (3.22) while employing the following substitution:

$$\cos\theta_t = \sqrt{1-\sin^2\theta_t} = i\sqrt{\frac{n_i^2}{n_t^2}\sin^2\theta_i - 1} \qquad (\theta_i > \theta_c) \tag{3.39}$$

(see P0.19).

In this case, θ_t is a complex number. However, we do not assign geometrical significance to it in terms of any direction. Actually, we don't even need to know the value for θ_t; we need only the values for $\sin\theta_t$ and $\cos\theta_t$, as specified by Snell's law (3.7) and (3.39). Even though $\sin\theta_t$ is greater than one and $\cos\theta_t$ is imaginary, we can use their values to compute r_s, r_p, t_s, and t_p. (Complex notation is wonderful!)

Upon substitution of (3.39) into the Fresnel reflection coefficients (3.20) and (3.22) we obtain

$$r_s = \frac{n_i\cos\theta_i - i n_t\sqrt{\frac{n_i^2}{n_t^2}\sin^2\theta_i - 1}}{n_i\cos\theta_i + i n_t\sqrt{\frac{n_i^2}{n_t^2}\sin^2\theta_i - 1}} \qquad (\theta_i > \theta_c) \tag{3.40}$$

[5] M. Born and E. Wolf, *Principles of Optics*, 7th ed., Sect. 1.5.4 (Cambridge University Press, 1999).