Final Exam 2014: Quantum Mechanics I

PART A

SOLUTION

An electron is inside an infinite well of length $L$ as shown here.

\[ \psi_n(x) = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{L} \right), \quad x \in [0, L] \] (1)

with energies $E_n = \frac{n^2 \hbar^2 \pi^2}{2mL^2}$. Q’s 1-8 pertain this system.

1. Choose which of the following wavefunctions $\Psi(x, 0)$ are allowed at $t = 0$.
   (a) I only.
   (b) I and II only.
   (c) I, II and III only.
   (d) all of them.
   (e) none of the above.
Answer 1:
The wavefunction in IV is discontinuous at $x = L/2$ and hence forbidden. Hence (c) option is the correct answer.

2. The electron is in the first excited state. What can you say about its position and momentum?

(a) $\langle x \rangle = 0$ and $\langle p \rangle = 0$.

(b) $\langle x \rangle = L/2$ and $\langle p \rangle = 0$.

(c) $\langle x \rangle = L/2$ and $p$ is totally uncertain, so $\langle p \rangle$ cannot be defined.

(d) $\langle x \rangle = 0$ and $\langle p \rangle = \hbar/L$.

(e) $\langle x \rangle = L/2$ and $\langle p \rangle = \hbar/L$.

Answer 2:
For a electron in first excited state ($n = 2$), in an infinite well of length $L$, the value of $\langle p \rangle$ is

$$\langle p \rangle = \langle \psi_2(x) | \hat{p} | \psi_2(x) \rangle$$

$$= \langle \psi_2(x) | \left( -i \hbar \frac{\partial}{\partial x} \right) | \psi_2(x) \rangle$$
\begin{align*}
&= -i\hbar \int_0^L dx \left( \sqrt{\frac{2}{L}} \sin \left( \frac{2\pi x}{L} \right) \right) \frac{\partial}{\partial x} \left( \sqrt{\frac{2}{L}} \sin \left( \frac{2\pi x}{L} \right) \right) \\
&= -i\hbar \int_0^L dx \sin \left( \frac{2\pi x}{L} \right) \left( \frac{2\pi}{L} \right) \\
&= -i\hbar \frac{2\pi}{L} \int_0^L dx \sin \left( \frac{4\pi x}{L} \right) \\
&= i\hbar \frac{2\pi}{L} \cos \left( \frac{4\pi x}{L} \right) \bigg|_0^L = \frac{i\hbar}{2L} (1 - 1) = 0,
\end{align*}

and expectation value of position is
\[
\langle x \rangle = \langle \psi_2(x) | \hat{x} | \psi_2(x) \rangle \\
= \frac{2}{L} \int_0^L dx \, x \sin^2 \left( \frac{2\pi x}{L} \right) \\
= \frac{1}{L} \int_0^L dx \left[ 1 - \cos \left( \frac{4\pi x}{L} \right) \right] \\
= \frac{1}{L} \left[ \int_0^L dx \, x - \int_0^L dx \, x \cos \left( \frac{4\pi x}{L} \right) \right] = \frac{1}{L} \left. \frac{x^2}{2} \right|_0^L = \frac{L}{2}.
\]

Option (b) is the correct answer. Infact, there is no need to perform this calculation as \( \langle x \rangle = L/2 \) is clear by inspection and \( \langle p \rangle = 0 \) since the wavefunction is a standing wave which is a superposition of two counter-propagating waves.

3. The electron is in the first excited state. Which of the following expectation values will change with time?

(a) \( \langle x \rangle \) only.

(b) \( \langle p \rangle \) only.

(c) \( \langle E \rangle \) only.

(d) all of \( \langle x \rangle, \langle p \rangle \) and \( \langle E \rangle \).

(e) neither of \( \langle x \rangle, \langle p \rangle \) and \( \langle E \rangle \) will change with time.

**Answer 3:**
Since \( \langle x \rangle \) and \( \langle p \rangle \) found in Q2, are time-independent and \( E \) is conserved. So option (e) is the correct answer.

4. Consider the wavefunction \( \Psi(x,0) = \sqrt{\frac{2}{L}} \sin \left( \frac{5\pi x}{L} \right) \). Which of the following is the probability density at some later time \( t \)?
(a) $\frac{2}{L} \sin^2 \left(\frac{5\pi x}{L}\right) \cos^2 \left(\frac{E_5 t}{\hbar}\right)$.
(b) $\frac{2}{L} \sin^2 \left(\frac{5\pi x}{L}\right)$.
(c) $\frac{2}{L} \sin^2 \left(\frac{5\pi x}{L}\right) e^{-i2E_5 t/\hbar}$.
(d) $\frac{2}{L} \sin^2 \left(\frac{5\pi x}{L}\right) e^{-iE_5 t/\hbar}$.
(e) $\frac{2}{L} \sin \left(\frac{5\pi x}{L}\right) \cos \left(\frac{5\pi x}{L}\right) e^{-2iE_5 t/\hbar}$.

**Answer 4:**

For an electron in an infinite potential well described by wavefunction

$$
\Psi(x, 0) = \sqrt{\frac{2}{L}} \sin \left(\frac{5\pi x}{L}\right),
$$

the initial quantum state in the Hilbert space is

$$
|\psi(0)\rangle = |\psi_5\rangle.
$$

At time $t$, the state becomes

$$
|\psi(t)\rangle = e^{-i\mathcal{H}t/\hbar} |\psi(0)\rangle = e^{-iE_5 t/\hbar} |\psi_5\rangle
$$

and

$$
\psi(t) = \langle x |\psi(t)\rangle = e^{-iE_5 t/\hbar} \psi_5(x).
$$

⇒ Probability density = $|\psi(x)|^2 = \frac{2}{L} \sin^2 \left(\frac{5\pi x}{L}\right)$,

option (b) is the correct answer.

5. The electron is known to have the wavefunction

$$
\Psi(x, 0) = \sqrt{\frac{5}{7}} \psi_1(x) + \sqrt{\frac{2}{7}} \psi_2(x),
$$

at time $t = 0$. Here $\psi_1(x)$ and $\psi_2(x)$ are the wavefunctions for the ground and first excited state. The energy measurement at $t = 0$ yields the outcome ($4\pi^2 \hbar^2 / 2mL^2$).

Which of the following is true?

(a) The normalized state after measurement is $\psi_1(x, 0)$ and will remain $\psi_1(x, 0)$ for all later times.
(b) The normalized state after measurement is $\psi_1(x, 0)$ and will evolve to

$$
\sqrt{\frac{5}{7}} \psi_1(x) + \sqrt{\frac{2}{7}} \psi_2(x)
$$
if one waits long enough.

(c) The state is completely annihilated and the electron disappears.

(d) The normalized state after measurement is $\psi_2(x)$ and continues to remain so for all later times.

(e) The normalized state after measurement is $\psi_2(x)$ and oscillates between $\psi_2(x)$ and $\psi_1(x)$ with time.

Answer 5:
Correct option is (d).

6. For the electron possessing the $t = 0$ wavefunction
\[
\Psi(x, 0) = \sqrt{\frac{5}{7}} \psi_1(x) + i \sqrt{\frac{2}{7}} \psi_2(x);
\]  
what can we say about the energy measurement? See the following statements. Note the presence of $i$ in the above.

(a) The energy measurement at $t > 0$ will always yield $\left(\frac{5}{7} E_1 + \frac{2}{7} E_2\right)$.

(b) The energy measurement at $t > 0$ will always yield $\left(\frac{5}{7} E_1 - \frac{2}{7} E_2\right)$.

(c) The energy measurement at $t > 0$ can yield any of $E_1, E_2, E_3, E_4, \ldots$, but the probability of measuring $E_1$ remains the highest.

(d) The energy measurement at $t > 0$ will either yield $E_1$ or $E_2$.

(e) none of the above is valid.

Answer 6:
For a particle possessing the $t = 0$ wavefunction (Eq (4)), the initial quantum state in the Hilbert space is the superposition of two quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$;
\[
|\psi(0)\rangle = \sqrt{\frac{5}{7}} |\psi_1\rangle + i \sqrt{\frac{2}{7}} |\psi_2\rangle;
\]
and at time $t > 0$, the state becomes
\[
|\psi(t)\rangle = \sqrt{\frac{5}{7}} e^{-iE_1t/\hbar} |\psi_1\rangle + i \sqrt{\frac{2}{7}} e^{-iE_2t/\hbar} |\psi_2\rangle
\]
Option (d) is correct.
7. For the wavefunction in Q(6), the expectation value of the energy at time $t$ is:

(a) $\frac{5}{7}E_1 + \frac{2}{7}E_2$.
(b) $\frac{5}{7}E_1 - \frac{2}{7}E_2$.
(c) $\frac{5}{7}E_1 e^{-iE_1t/\hbar} + \frac{2}{7}E_2 e^{-iE_2t/\hbar}$.
(d) $\frac{5}{7}E_1 e^{-iE_1t/\hbar} - \frac{2}{7}E_2 e^{-iE_2t/\hbar}$.
(e) $\frac{E_1 + E_2}{2}$.

**Answer 7:**

To find the expectation of $E$, first we need to find $\Psi(x,t)$, which from Q6 is

$$\Psi(x,t) = \sqrt{\frac{5}{7}} e^{-iE_1t/\hbar} \psi_1(x) + i \sqrt{\frac{2}{7}} e^{-iE_2t/\hbar} \psi_2(x).$$

Now $\langle E \rangle = \langle \Psi(x,t) | \hat{H} | \Psi(x,t) \rangle$

$$= \int_0^L dx \left( \sqrt{\frac{5}{7}} \psi_1(x) e^{iE_1t/\hbar} - i \sqrt{\frac{2}{7}} \psi_2(x) e^{iE_2t/\hbar} \right) \hat{H} \left( \sqrt{\frac{5}{7}} \psi_1(x) e^{-iE_1t/\hbar} + i \sqrt{\frac{2}{7}} \psi_2(x) e^{-iE_2t/\hbar} \right)$$

$$= \frac{1}{7} \left[ \int_0^L dx \left( \sqrt{5} \psi_1(x) e^{iE_1t/\hbar} - i \sqrt{2} \psi_2(x) e^{iE_2t/\hbar} \right) \left( \sqrt{5} E \psi_1(x) e^{-iE_1t/\hbar} + i \sqrt{2} E \psi_2(x) e^{-iE_2t/\hbar} \right) \right]$$

$$= \frac{1}{7} \left[ 5E_1 \int_0^L |\psi_1(x)|^2 dx + 2E_2 \int_0^L |\psi_2(x)|^2 dx + 0 \right],$$

where

$$\int_0^L \psi_1(x) \psi_2(x) dx = \frac{2}{L} \int_0^L \sin \left( \frac{\pi x}{L} \right) \sin \left( \frac{2\pi x}{L} \right) dx = 0.$$

So the expectation value of $E$ is

$$\langle E \rangle = \frac{5}{7}E_1 + \frac{2}{7}E_2.$$

Option (a) is correct.

8. The position of the particle in the infinite well in the state $\frac{1}{\sqrt{2}} (|\psi_1\rangle + |\psi_2\rangle)$ is measured to be $x_0$. Immediately after the position measurement, its energy is measured. What is the outcome?

(a) The measured energy is $\frac{E_1 + E_2}{2}$.  

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(b) We can only measure either \( E_1 \) or \( E_2 \).

(c) We will obtain one of the energy values \( E_n = \frac{n^2 \hbar^2 \pi^2}{2mL^2} \), where \( n \) can be any positive integer.

(d) We will measure the energy

\[
\sum_{n=1}^{\infty} c_n E_n, \quad \text{where} \quad \sum_{n=1}^{\infty} |c_n|^2 = 1.
\]

(e) none of the above.

**Answer 8:**
Measurement \( x_0 \) indicates the creation of the state \( \psi(x) = \delta(x - x_0) \) which in the momentum representation is \( \tilde{\psi}(p) \propto e^{ipx_0/\hbar} \) with all possible values of \( p \) allowed. Hence \( E \) can take up all allowed values given by \( n^2 \hbar^2 \pi^2 / 2mL^2 \). Option (c) is correct.

9. A Cs atom bounces off a perfect mirror (shown overleaf). Such experiments are now realizable. We are interested in predicting the time-independent wavefunction \( \psi(z) \) where \( z \) is the vertical distance from the mirror held at zero potential. The potential energy is gravitational. Which of the following graphs shows an allowed form of \( \text{Re} \{ \psi(z) \} \). Take \( V(z < 0) \rightarrow \infty \).
Answer 9:

By increasing $z$, the potential energy $mgz$ increases, so the wave vector $k = \sqrt{\frac{2m}{\hbar^2}(E - V)}$ decreases and wavelength $\lambda = 2\pi/k$ increases. Furthermore the wavefunction damps. Option (e) is correct.

10. An $^{55}$Fe nucleus is spin-$3/2$. The matrix representation of $\hat{S}_x$ for such a particle in the
\( \hat{S}_z \) basis is given by

\[
\hat{S}_z = \frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix}
\]

where the elements are ordered as \{ |\frac{3}{2}, \frac{3}{2}\>, |\frac{3}{2}, \frac{1}{2}\>, |\frac{3}{2}, -\frac{1}{2}\>, |\frac{3}{2}, -\frac{3}{2}\> \}.

Complete the following table. 

\[ \begin{array}{|c|c|}
\hline
\text{Input} & \text{Fraction of nuclei transmitted} \\
\hline
\text{State is } |\frac{3}{2}, \frac{3}{2}\>. & 1/8 \\
\hline
\text{State is an equal superposition} & \frac{1}{32} (2 + 2\sqrt{3})^2 \approx 0.933. \\
\text{of all eigenstates of } \hat{S}_z. & \\
\hline
\text{A stream of many } ^{55}\text{Fe nuclei} & 1/4 \\
\text{comprising an equal mixture of four} & \\
\text{eigenstates, which means 25\% of} & \\
\text{each eigenstate of } \hat{S}_z. & \\
\hline
\end{array} \]

**Answer 10:**

To find the fraction of nuclei transmitted when a measurement perform on \(^{55}\text{Fe}\) in state \(|\psi\rangle = |\frac{3}{2}, \frac{3}{2}\>\) yields \(S_z = 3\hbar/2\), we need to prepare an eigenstate of \(\hat{S}_z\) corresponding to eigenvalue \(+3/2\). The eigenvalue equation of \(\hat{S}_z|v\rangle = m_s \hbar |v\rangle\) with \(m_s = 3/2\) is

\[
\frac{\hbar}{2} \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix} = \frac{3\hbar}{2} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix},
\]
from which we discover,

\[ \sqrt{3}e_2 = 3e_1 \]
\[ \sqrt{3}e_1 + 2e_3 = 3e_2 \]
\[ 2e_2 + \sqrt{3}e_4 = 3e_3 \]
\[ \sqrt{3}e_3 = 3e_4 \]

By setting \( e_1 = 1 \) and solving these equations simultaneously, we obtain

\[ e_1 = 1, \ e_2 = \sqrt{3}, \ e_3 = \sqrt{3}, \ e_4 = 1. \]

So the normalized eigenstate of \( \hat{S}_x \) with eigenvalue \( 3\hbar/2 \) is

\[ |v\rangle = \frac{1}{\sqrt{8}} \begin{pmatrix} 1 \\ \sqrt{3} \\ \sqrt{3} \\ 1 \end{pmatrix}. \]

Therefore, the fraction of nuclei transmitted is

\[ |\langle v|\psi \rangle|^2 = \left| \frac{1}{\sqrt{8}} \begin{pmatrix} 1 & \sqrt{3} & \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right|^2 = 1/8. \]

Similarly, if the input state is an equal superposition of all eigenstates of \( \hat{S}_z \);

\[ |\psi\rangle = \frac{1}{\sqrt{4}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \]

then the fraction of transmitted nuclei is

\[ \langle v|\psi \rangle = \frac{1}{\sqrt{8} \sqrt{4}} \begin{pmatrix} 1 & \sqrt{3} & \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{32}} (2 + 2\sqrt{3}) \]

\[ \Rightarrow |\langle v|\psi \rangle|^2 = \frac{1}{32} (2 + 2\sqrt{3})^2 \approx 0.933. \]
If the input is a stream of Fe nuclei in different states, i.e., 25% of each eigenstate of $\hat{S}_z$, then $\frac{1}{4}$ of the nuclei will be transmitted.
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PART B

SOLUTION

Attempt all questions.

1. A free particle has the wavefunction $\psi(x) = A(e^{ikx} + e^{-ikx})$, $k$ being the wave vector.

(a) Plot the real part of $\psi(x)$ versus $x$. [2]

(b) Derive and plot the real and imaginary parts of the wavefunction in the momentum space $\tilde{\psi}(p)$. [5]

Use

$$\frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dx \, e^{-i(p-p')x/\hbar} = \delta(p - p')$$

**Answer**

(a)
(b) The wavefunction in momentum space is defined as

\[ \tilde{\psi}(p) = \langle p|\psi \rangle \]
\[ = \langle p|\hat{1}|\psi \rangle \]
\[ = \int_{-\infty}^{\infty} dx \langle p|x \rangle \langle x|\psi \rangle \]
\[ = \int_{-\infty}^{\infty} dx \psi^\dagger_p(x)\psi(x), \]

where \( \psi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar} \), is the normalized wavefunction of the momentum eigenstate \( |p\rangle \) in position basis.

\[ \tilde{\psi}(p) = \frac{A}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx e^{-ipx/\hbar}(e^{ikx} + e^{-ikx}) \]
\[ = \frac{A}{\sqrt{2\pi\hbar}} \left[ \int_{-\infty}^{\infty} dx e^{-i(p-k)x/\hbar} + \int_{-\infty}^{\infty} dx e^{-i(p+k)x/\hbar} \right] \]
\[ = \frac{A}{\sqrt{2\pi\hbar}} [2\pi\hbar \delta(p - \hbar k) + 2\pi\hbar \delta(p + \hbar k)] \quad \text{using the provided integral} \]
\[ = \sqrt{2\pi\hbar} A [\delta(p - \hbar k) + \delta(p + \hbar k)]. \]

and \( \text{Im} \tilde{\psi}(p) = 0. \)

These are two Dirac delta functions at \( p=\pm \hbar k \)

2. The solar neutrino problem was a long outstanding problem in physics.

Today, we know that neutrinos are highly relativistic particles with nonzero mass. Neutrinos can occur in two mutually orthogonal states
labeled as $|v_e\rangle$ and $|v_\mu\rangle$ corresponding to the two ‘flavors’ of neutrinos—
called the electron neutrino and muon neutrino respectively. These states are eigenstates of the weak interaction Hamiltonian. However, when neutrinos propagate in free space, the only Hamiltonian of relevance is due to relativistic energy of the particles. The eigenstates of this Hamiltonian are generally called the mass eigenstates, denoted by $|v_1\rangle$ and $|v_2\rangle$.

Hence states in the two dimensional Hilbert space can be described by the weak interaction eigenstates $\{ |v_e\rangle, |v_\mu\rangle \}$ or by the mass eigenstates $\{ |v_1\rangle, |v_2\rangle \}$. The relationship between the states is described by

$$|v_e\rangle = \cos\frac{\theta}{2} |v_1\rangle + \sin\frac{\theta}{2} |v_2\rangle$$ (3)

$$|v_\mu\rangle = \sin\frac{\theta}{2} |v_1\rangle - \cos\frac{\theta}{2} |v_2\rangle$$ (4)

The angle $\theta/2$ is called the mixing angle.

(a) Write the similarity matrix that takes a state vector from the $\{ |v_e\rangle, |v_\mu\rangle \}$ (weak basis) to the $\{ |v_1\rangle, |v_2\rangle \}$ (mass basis). [3]

(b)
Suppose an electron neutrino $|\psi(0)\rangle = |v_e\rangle$ is created on the sun. In its free propagation towards the earth, only the mass Hamiltonian is operative. The eigenvalues of this Hamiltonian are $E_1$ and $E_2$ for the eigenstates $|v_1\rangle$ and $|v_2\rangle$. Write the time-dependent quantum state $|\psi(t)\rangle$ as neutrinos propagate in space \[3\]

(c) Suppose it takes time $\tau$ for the neutrino to reach a detector on the earth. The detector can only see muon neutrinos $|v_\mu\rangle$. Find the probability $P$ that the detector detects a muon neutrino, when it is directly facing the stream of neutrinos. (Assume no neutrinos are lost in between.) \[5\]

(d) Neutrinos are relativistic particles. This means that the mass eigenstates have energies

$$E_1 = \sqrt{(m_1c^2)^2 + (pc)^2}$$
$$E_2 = \sqrt{(m_2c^2)^2 + (pc)^2},$$

where $m_{1,2}$ are masses of the mass eigenstates, $c$ is the speed of light and $p$ is their common momentum. For relativistic particles, $(pc) \gg (mc^2)$. Using Binomial expansion show that \[4\]

$$E_1 - E_2 = \frac{c^3}{2p}(m_1^2 - m_2^2).$$

(e) Using the above expression for $E_1 - E_2$, find the probability $P_{v_e\rightarrow v_\mu}$ found in (c), expressed in terms of the distance $L$. \[2\]

(f) How far will an 8 MeV electron neutrino travel before being converted into a muon neutrino?
Assume, $m_1^2 - m_2^2 \approx 8 \times 10^{-5} \text{eV}^2/c^4$ and use $p = E/c$ to calculate the momentum of the relativistic particle. \[3\] 

**Answer**

(a) A similarity matrix $\hat{S}$ that takes a state vector from the $\{ |v_e\rangle, |v_\mu\rangle \}$ (weak basis) to the $\{ |v_1\rangle, |v_2\rangle \}$ (mass basis) is given by 

$$
\hat{S} = \begin{pmatrix}
\langle v_1 | v_e \rangle & \langle v_1 | v_\mu \rangle \\
\langle v_2 | v_e \rangle & \langle v_2 | v_\mu \rangle 
\end{pmatrix} = \begin{pmatrix}
\hat{S}_{11} & \hat{S}_{12} \\
\hat{S}_{21} & \hat{S}_{22}
\end{pmatrix}.
$$

Using Eqs (3) and (4), the matrix elements are 

$$
\hat{S}_{11} = \cos \frac{\theta}{2}, \quad \hat{S}_{12} = \sin \frac{\theta}{2}, \\
\hat{S}_{21} = \sin \frac{\theta}{2}, \quad \hat{S}_{22} = -\cos \frac{\theta}{2}.
$$

Hence the similarity matrix $\hat{S}$ is 

$$
\hat{S} = \begin{pmatrix}
\cos \theta/2 & \sin \theta/2 \\
\sin \theta/2 & -\cos \theta/2
\end{pmatrix}.
$$

(b) The initial quantum state of an electron neutrino created on the sun is 

$$
|\psi(0)\rangle = |v_e\rangle = \cos \frac{\theta}{2} |v_1\rangle + \sin \frac{\theta}{2} |v_2\rangle.
$$

At time $t$, the state becomes 

$$
|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = e^{-i\hat{H}t/\hbar} \left( \cos \frac{\theta}{2} |v_1\rangle + \sin \frac{\theta}{2} |v_2\rangle \right).
$$

Here $\hat{H}$ is the mass Hamiltonian with eigenstates $|v_1\rangle$ and $|v_2\rangle$ and eigenvalues $E_1$ and $E_2$ 

$$
|\psi(t)\rangle = \cos \frac{\theta}{2} e^{-iE_1t/\hbar} |v_1\rangle + \sin \frac{\theta}{2} e^{-iE_2t/\hbar} |v_2\rangle.
$$
(c) The probability of detecting a muon neutrino at time \( t = \tau \) is

\[
P(\mu, \tau) = \left| \langle v_\mu | \psi(\tau) \rangle \right|^2
\]

\[
\langle v_\mu | \psi(\tau) \rangle = \left( \sin \frac{\theta}{2} \langle v_2 | - \cos \frac{\theta}{2} \langle v_2 | \right) \left( \cos \frac{\theta}{2} e^{-iE_1 \tau / \hbar} | v_1 \rangle + \sin \frac{\theta}{2} e^{-iE_2 \tau / \hbar} | v_2 \rangle \right)
\]

\[
= \sin \frac{\theta}{2} \cos \frac{\theta}{2} e^{-iE_1 \tau / \hbar} - \sin \frac{\theta}{2} \cos \frac{\theta}{2} e^{-iE_2 \tau / \hbar}
\]

\[
= \frac{1}{2} \sin \theta (e^{-iE_1 \tau / \hbar} - e^{-iE_2 \tau / \hbar}),
\]

where we have used the half angle formula \( \sin \theta = 2 \sin(\theta/2) \cos(\theta/2) \).

\[
\Rightarrow \left| \langle v_\mu | \psi(\tau) \rangle \right|^2 = \frac{1}{4} \sin^2 \theta \left( e^{iE_1 \tau / \hbar} - e^{iE_2 \tau / \hbar} \right) \left( e^{-iE_1 \tau / \hbar} - e^{-iE_2 \tau / \hbar} \right)
\]

\[
= \frac{1}{4} \sin^2 \theta \left( 1 + 1 - e^{i(E_1 - E_2) \tau / \hbar} - e^{-i(E_1 - E_2) \tau / \hbar} \right)
\]

\[
= \frac{1}{4} \sin^2 \theta \left( 2 - 2 \cos \left( \frac{(E_1 - E_2) \tau}{\hbar} \right) \right)
\]

\[
= \frac{1}{2} \sin^2 \theta \left( 1 - \cos \left( \frac{(E_1 - E_2) \tau}{\hbar} \right) \right)
\]

\[
= \sin^2 \theta \sin^2 \left( \frac{(E_1 - E_2) \tau}{2\hbar} \right),
\]

where \( \sin^2(\theta/2) = \frac{1 - \cos \theta}{2} \).

(d) The energy corresponds to mass eigenstate \( |v_1\rangle \) is given by

\[
E_1 = \sqrt{((pc)^2 + m_1 c^2)^2}
\]

\[
= (pc) \sqrt{1 + \left( \frac{m_1 c^2}{(pc)^2} \right)^2}.
\]

For relativistic particles, \( (mc^2)/(pc) \ll 1 \). So by using Binomial expansion,

\[
E_1 \approx (pc) \left( 1 + \frac{(m_1 c^2)^2}{2(pc)^2} \right)
\]

\[
= pc + \frac{m_1^2 c^4}{2p}.
\]
Similarly,

\[ E_2 = pc + \frac{m_2^2c^3}{2p} \]

\[ E_1 - E_2 = \frac{m_1^2c^3}{2p} - \frac{m_2^2c^3}{2p} = \frac{c^3}{2p}(m_1^2 - m_2^2). \]

(e) From part (c) and (d),

\[ P_{v_e \rightarrow v_\mu} = \sin^2 \theta \sin^2 \left( \frac{c^3}{2p} \left( m_1^2 - m_2^2 \right) \frac{\tau}{2\hbar} \right) \]

\[ = \sin^2 \theta \sin^2 \left( \frac{c^2(m_1^2 - m_2^2)c\tau}{4ph} \right) \]

\[ = \sin^2 \theta \sin^2 \left( \frac{c^2(m_1^2 - m_2^2)L}{4ph} \right) \quad \text{using} \quad L \approx c\tau \]

(f) When the argument of the \( \sin^2 \) function changes by \( \pi/2 \), the probability goes from zero to one. Hence one can find the distance as

\[ \frac{c^2(m_1^2 - m_2^2)\Delta L}{4ph} = \frac{\pi}{2} \]

\[ \Delta L = \frac{2\pi ph}{c^2(m_1^2 - m_2^2)} \]

\[ = \frac{ph}{c^2(m_1^2 - m_2^2)} \]

\[ = \frac{Eh}{c^3(m_1^2 - m_2^2)} \quad \text{using} \quad p = E/c \]

\[ = \frac{(8 \times 10^6)\left(1.602 \times 10^{-19}\right)(6.63 \times 10^{-34})}{(3 \times 10^8)^3(8 \times 10^{-5})(1.602 \times 10^{-19})^2/(3 \times 10^8)^2} \]

\[ \approx 124 \text{ km.} \]

3. Field emission is a common technique to eject electrons from a metal surface. In a metal, electrons are filled upto the Fermi level \( E_F \).
The minimum energy required to eject the electron from the metal is $W$, called the work function. If a nearby metal is placed at a positive voltage $V_0$ w.r.t. the bulk metal tip, an electric field is setup. As a result the potential energy profile changes as shown in figure (c). The Fermi electron can now easily tunnel out of the metal tip.

(a) If $|\mathcal{E}|$ is the constant electric field between the tip and the nearby metal, express $V(x) - E_F$ in terms of the electric field. Use the notation $[3]$

$V(x) =$ potential energy, $v =$electric voltage,

$E_F =$Fermi energy = energy of tunneling electron.

(b) Show that the tunneling property of the Fermi electron is

$$T \approx \exp\left(\frac{-4\sqrt{2mW^{3/2}}}{3e|\mathcal{E}|\hbar}\right).$$

Use the approximate approximation

$$T \approx \exp\left(\frac{-2\sqrt{2m}}{\hbar} \int_a^b dx \sqrt{V(x) - E}\right),$$

choosing the approximate integration limits. If you encounter an integral of the form $\int dz \sqrt{1 - z^2}$, make the substitution $z = \cos \theta$.

$[7]$
(c) Make a graph between $\ln T$ and $x$. Identify the slope of the graph.

[3]

Answer

(a) From the figure (c), it is clear that the potential energy varies in the region $x > 0$ as

\[
V(x) = E_F + W - e|\mathcal{E}|x
\]

\[
V(x) - E_F = W - e|\mathcal{E}|x.
\]

(b) The approximate expression for tunneling coefficient is given by

\[
T \approx \exp\left(\frac{-2\sqrt{2m}}{\hbar} \int_a^b dx \sqrt{V(x) - E}\right).
\]

First we have to calculate the integral,

\[
I = \int_a^b dx \sqrt{V(x) - E}
\]

\[
= \int_a^b dx \sqrt{W - e|\mathcal{E}|x}
\]

\[
= \sqrt{W} \int_a^b dx \sqrt{1 - \frac{e|\mathcal{E}|x}{W}}.
\]

Here $[a, b]$ corresponds to classically forbidden region (or tunneling allowed region). So it is clear that $a = 0$, $b$ is the value of $x$ when $V(x) = E_F$, i.e.,

\[
W - e|\mathcal{E}|b = 0, \quad \Rightarrow b = \frac{W}{e|\mathcal{E}|}.
\]

So we have

\[
I = \sqrt{W} \int_0^{\frac{W}{e|\mathcal{E}|}} dx \sqrt{1 - \frac{e|\mathcal{E}|x}{W}}.
\]
Let
\[ \frac{e|\mathcal{E}|x}{W} = z^2, \quad \Rightarrow x = \frac{W}{e|\mathcal{E}|}z^2 \]
\[ \Rightarrow dx = \frac{2Wz}{e|\mathcal{E}|}dz. \]

When \( x = 0 \), \( z = 0 \) and when \( x = \frac{W}{e|\mathcal{E}|} \), \( z = 1 \). By these substitutions, integral becomes
\[ I = \sqrt{W} \int_{0}^{1} \left( \frac{2W}{e|\mathcal{E}|} \right) z\sqrt{1-z^2}dz. \]

Put \( z = \cos \theta \), \( \Rightarrow dz = -\sin \theta d\theta \). When \( z = 0 \), \( \theta = \pi/2 \) and when \( z = 1 \), \( \theta = 0 \). Therefore,
\[ I = \frac{2(W)^{3/2}}{e|\mathcal{E}|} \int_{0}^{\pi/2} \cos \theta \sin \theta (-\sin \theta d\theta) \]
\[ = \frac{2(W)^{3/2}}{e|\mathcal{E}|} \int_{0}^{\pi/2} \cos \theta \sin^2 \theta d\theta \]
\[ = \frac{2(W)^{3/2}}{e|\mathcal{E}|} \frac{\sin^3 \theta}{3} \bigg|_{0}^{\pi/2} = \frac{2(W)^{3/2}}{3e|\mathcal{E}|}. \]

By inserting it in tunneling expression,
\[ T \approx \exp \left( \frac{-2\sqrt{2m} 2(W)^{3/2}}{\hbar 3e|\mathcal{E}|} \right) \]
\[ = \exp \left( \frac{-4\sqrt{2m}(W)^{3/2}}{3e\hbar|\mathcal{E}|} \right), \]

which is the desired result.

(c) Discarded \( T \) is independent of \( x \). I actually wanted to ask for a plot between \( \ln T \) and \( |\mathcal{E}| \).

4. A molecular magnet is an organic molecule that behaves like a tiny magnet due to its spin. The structure of one such molecule is shown in the figure.
Consider a molecule with spin $s = 2$. The approximate Hamiltonian describing the time dynamics for this molecule is

$$\hat{H} = -\left(\frac{D}{\hbar}\right)\hat{S}_z^2,$$  \hspace{1cm} (5)

where $D$ is an anisotropy constant.

(a) Write the Hamiltonian in the Zeeman basis (basis of eigenstates of $\hat{S}_z$). \hspace{1cm} [4]

(b) What is the ground state energy of the molecule and what is its degeneracy? \hspace{1cm} [3]

(c) Using

$$\hat{S}_\pm |s, m_s\rangle = \sqrt{s(s+1) - m_s(m_s \pm 1)} \hbar |s, m_s \pm 1\rangle,$$  \hspace{1cm} (6)

find the matrix representation of $\hat{S}_+^2 + \hat{S}_-^2$. \hspace{1cm} [4]

(d) If the exact Hamiltonian is

$$\hat{\mathcal{H}} = -\left(\frac{D}{\hbar}\right)\hat{S}_z^2 + \left(\frac{A}{\hbar}\right)(\hat{S}_+^2 + \hat{S}_-^2),$$  \hspace{1cm} (7)

and initial state is $|s, m_s\rangle = |2, 1\rangle$, what is the state after time $t$. \hspace{1cm} [10]

HINT: First write the matrix form of Hamiltonian in Eq (7).
**Answer**

(a) A spin 2 molecule has $s = 2$, so there are $(2s + 1) = 2(2) + 1 = 5$ allowed values of the magnetic quantum number $m_s$

$$m_s = 2, 1, -0, -1, -2.$$  

and 5 eigenstates of $\hat{S}_z^2$ labeled as

$$|s, m_s\rangle = |2, 2\rangle, \ |2, 1\rangle, \ |2, 0\rangle, \ |2, -1\rangle, \ |2, -2\rangle.$$  

with eigenvalues of $\hat{S}_z^2$ being

$$h^2 m_s^2 = 4h^2, \ h^2, \ 0, \ + h^2, \ + 4h^2.$$  

So the eigenvalues of the Hamiltonian in Eq (5), are $-DHm_s^2$. It thus follows that

$$\langle s, m_s|\hat{H}|s, m_s\rangle = -DHm_s^2\langle s, m_s|s, m_s\rangle$$

$$= -DHm_s^2$$

$$\Rightarrow \langle 2, 2|\hat{H}|2, 2\rangle = -4Dh, \ \langle 2, 1|\hat{H}|2, 1\rangle = -Dh$$

$$\langle 2, 0|\hat{H}|2, 0\rangle = 0, \ \langle 2, -1|\hat{H}|2, -1\rangle = -Dh$$

$$\langle 2, -2|\hat{H}|2, -2\rangle = -4Dh,$$

and off-diagonal matrix elements of the Hamiltonian equal to zero.

Hence the matrix representation of $\hat{H}$ in the Zeeman basis is
\[
\hat{H} = \begin{pmatrix}
-4Dh & 0 & 0 & 0 & 0 \\
0 & -Dh & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -Dh & 0 \\
0 & 0 & 0 & 0 & -4Dh
\end{pmatrix}
\]

\[
= -Dh
\begin{pmatrix}
4 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & +1 & 0 \\
0 & 0 & 0 & 0 & +4
\end{pmatrix}.
\]

(b) For \(D\) is positive, ground state is \(\ket{2,2}\) and \(\ket{2,-2}\) with energy \(-4Dh\). It is a two-fold degenerate state.
If \(D\) is negative, then ground state \(\ket{2,0}\), which is one-fold degenerate.

(c) To find the matrix representation of \(\hat{S}_+^2 + \hat{S}_-^2\), first we need to know the matrix representations of \(\hat{S}_+\) and \(\hat{S}_-\).

Using Eq (6), we can find

\[
\hat{S}_+ \ket{2,2} = 0
\]

\[
\hat{S}_+ \ket{2,1} = \sqrt{2(2+1) - 1(1+1)} \ h \ket{2,2} = 2h \ket{2,2}
\]

\[
\hat{S}_+ \ket{2,0} = \sqrt{6}h \ket{2,1}
\]

\[
\hat{S}_+ \ket{2,-1} = \sqrt{6}h \ket{2,0}
\]

\[
\hat{S}_+ \ket{2,-2} = 2h \ket{2,-1}.
\]
From which we infer that in Zeeman basis,

$$\hat{S}_+ = \hbar \begin{pmatrix}
0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & \sqrt{6} & 0 & 0 \\
0 & 0 & \sqrt{6} & 0 \\
0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0
\end{pmatrix},$$

$$\hat{S}_- = \hat{S}_+^\dagger = \hbar \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{6} & 0 \\
0 & 0 & 0 & \sqrt{6} \\
0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

Therefore,

$$\hat{S}_+^2 = \hbar^2 \begin{pmatrix}
0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & \sqrt{6} & 0 & 0 \\
0 & 0 & \sqrt{6} & 0 \\
0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
0 & \sqrt{6} & 0 & 0 \\
0 & 0 & \sqrt{6} & 0 \\
0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0
\end{pmatrix}$$
\[
\hat{S}_-^2 = (\hat{S}_+^2)^\dagger = \hbar^2 \begin{pmatrix}
0 & 0 & 2\sqrt{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2\sqrt{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\hat{S}_+^2 + \hat{S}_-^2 = \hbar^2 \begin{pmatrix}
0 & 0 & 2\sqrt{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix} + \hbar^2 \begin{pmatrix}
0 & 0 & 2\sqrt{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[
= \hbar^2 \begin{pmatrix}
0 & 0 & 2\sqrt{6} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
(d) Exact Hamiltonian is

\[
\hat{H} = -D \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & +1 & 0 \\
0 & 0 & 0 & +4 & 0 \end{pmatrix} + A \begin{pmatrix} 0 & 0 & 2\sqrt{6} & 0 & 0 \\
0 & 0 & 0 & 6 & 0 \\
2\sqrt{6} & 0 & 0 & 0 & 2\sqrt{6} \\
0 & 6 & 0 & 0 & 0 \\
0 & 0 & 2\sqrt{6} & 0 & 0 \end{pmatrix}
\]

\[
= \hbar \begin{pmatrix} -4D & 0 & 2\sqrt{6}A & 0 & 0 \\
0 & -D & 0 & 6A & 0 \\
2\sqrt{6}A & 0 & 0 & 0 & 2\sqrt{6}A \\
0 & 6A & 0 & -D & 0 \\
0 & 0 & 2\sqrt{6}A & 0 & -4D \end{pmatrix}
\]

See the enclosed elements in the above matrix. The Hamiltonian connects \(|2, 1\rangle\) to \(|2, -1\rangle\) and vice versa. So if a system starts in \(|2, 1\rangle\), it can only evolve into \(|2, -1\rangle\). Hence we need to focus only at the elements shown above.

\[
\hat{H}_{\text{new}} = \hbar \begin{pmatrix} -D & 6A \\
6A & -D \end{pmatrix},
\]

where the ordering of elements is \(\{|2, 1\rangle, |2, -1\rangle\}\). Other states don’t matter if one starts in \(|2, 1\rangle\). We need to express \(|2, 1\rangle\) in the eigenstates of \(\hat{H}_{\text{new}}\).

The eigenvalues are determined by requiring

\[
\det(\hat{H}_{\text{new}} - \lambda \hat{1}) = 0
\]

\[
\begin{vmatrix}
-D - \lambda & 6A \\
6A & -D - \lambda
\end{vmatrix} = 0,
\]
which yields $\lambda = -D \pm 6A$.

We can now find the eigenstates corresponding to these eigenvalues. For $\lambda_1 = -D + 6A$,

$$\begin{pmatrix} -D & 6A \\ 6A & -D \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = (-D + 6A) \begin{pmatrix} e_1 \\ e_2 \end{pmatrix},$$

yielding

$$-De_1 + 6Ae_2 = (-D + 6A)e_1$$

$$6Ae_1 - De_2 = (-D + 6A)e_2.$$

from which we discover $e_1 = e_2$. By setting $e_1 = 1$ we obtain

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Likewise for the other eigenstates, $\lambda_2 = -D - 6A$, we find

$$\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

We can normalize these states to obtain the eigenstates,

$$|v_1\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ with eigenvalue } -D + 6A, \text{ and}$$

$$|v_2\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \text{ with eigenvalue } -D - 6A.$$

In the small subspace spanned by $\{|2,1\rangle, |2,-1\rangle\}$ basis, our initial state is

$$|2,1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} (|v_1\rangle + |v_2\rangle).$$
Hence, with time the state evolves to

\[
|\psi(t)\rangle = \frac{1}{\sqrt{2}} \left( e^{-i\lambda_1 t} |v_1\rangle + e^{-i\lambda_2 t} |v_2\rangle \right) \\
= \frac{1}{\sqrt{2}} \left( e^{-i(-D+6A)t} |v_1\rangle + e^{-i(-D-6A)t} |v_2\rangle \right) \\
= \frac{1}{\sqrt{2}} \left( e^{iDt} e^{-i6At} |v_1\rangle + e^{iDt} e^{i6At} |v_2\rangle \right) \\
= \frac{1}{\sqrt{2}} e^{iDt} \left( e^{-i6At} |v_1\rangle + e^{i6At} |v_2\rangle \right) \\
= \frac{1}{2} e^{iDt} \left[ e^{-i6At} \left( |2, 1\rangle + |2, -1\rangle \right) + e^{i6At} \left( |2, 1\rangle - |2, -1\rangle \right) \right] \\
= \frac{1}{2} e^{iDt} \left[ \left( e^{i6At} + e^{-i6At} \right) |2, 1\rangle - \left( e^{i6At} - e^{-i6At} \right) |2, -1\rangle \right] \\
= \frac{1}{2} e^{iDt} \left[ 2 \cos(6At) |2, 1\rangle - 2i \sin(6At) |2, -1\rangle \right].
\]