Assignment 2: Solution

1. A measurement of $\hat{\varphi}_{HV}$ is performed for a beam of photons prepared in the state

$$|\psi\rangle = \frac{1}{\sqrt{3}}|H\rangle + \sqrt{\frac{2}{3}} e^{i\pi/3}|V\rangle.$$ 

What are the possible outcomes of this measurement? What are their probabilities? For each outcome, what state is the system left in?

Here $+1$ and $-1$ are measurement outcomes from non-demolitive measurement of the photon polarization in the $\{|H\rangle, |V\rangle\}$ basis.

Answer

The possible measurement outcomes are the eigenvalues of $\hat{\varphi}_{HV}$: $+1$ for horizontal, and $-1$ for vertical.

For a measurement yielding $+1$, the probability is

$$P\left(+1\Big|\psi\right) = |\langle H|\psi\rangle|^2$$

$$\langle H|\psi\rangle = \langle H\left(\frac{1}{\sqrt{3}}|H\rangle + \sqrt{\frac{2}{3}} e^{i\pi/3}|V\rangle\right) = \frac{1}{\sqrt{3}}$$

$$|\langle H|\psi\rangle|^2 = \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3}.$$ 

After such a measurement, the beam of photons is left in the state $|H\rangle$.

For a measurement yielding $-1$, the probability is

$$P\left(-1\Big|\psi\right) = |\langle V|\psi\rangle|^2$$

Now $\langle V|\psi\rangle = \langle V\left(\frac{1}{\sqrt{3}}|H\rangle + \sqrt{\frac{2}{3}} e^{i\pi/3}|V\rangle\right) = \sqrt{\frac{2}{3}} e^{i\pi/3}$

$\therefore |\langle V|\psi\rangle|^2 = \left(\sqrt{\frac{2}{3}} e^{-i\pi/3}\right)\left(\sqrt{\frac{2}{3}} e^{i\pi/3}\right) = \frac{2}{3}.$

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You can readily check that the probabilities add up to one. With such a measurement outcome, the beam of photons is left in the state $|V\rangle$.

2. (a) In analogy with $\hat{\gamma}_{HV}$ and $\hat{\gamma}_{45}$, define $\hat{\gamma}_C$, which is the operator corresponding to measurement of circular polarization. Define this operator such that measurements of left-circular polarization yield positive values. Find the matrix representation of $\hat{\gamma}_C$ in the $HV$-basis.

(b) A measurement of $\varphi_{HV}$ is performed on a photon prepared in state $|L\rangle$. A measurement of $\varphi_C$ (defined in part (a)) is then performed on a second photon prepared in the same state. What is the probability that the first measurement returns +1, and the second returns $-1$?

**Answer**

(a) The Hermitian operator $\hat{\gamma}_C$ corresponding to observation of circular polarization can be defined as

$$\hat{\gamma}_C|L\rangle = (+1)|L\rangle, \quad \hat{\gamma}_C|R\rangle = (-1)|R\rangle.$$

Another way to express $\hat{\gamma}_C$ is through the spectral decomposition

$$\hat{\gamma}_C = (+1)|L\rangle\langle L| + (-1)|R\rangle\langle R|.$$

In the $\{|H\rangle, |V\rangle\}$ basis, the matrix representation of $\hat{\gamma}_C$ is given by

$$\hat{\gamma}_C = \begin{pmatrix} \langle H|\hat{\gamma}_C|H\rangle & \langle H|\hat{\gamma}_C|V\rangle \\ \langle V|\hat{\gamma}_C|H\rangle & \langle V|\hat{\gamma}_C|V\rangle \end{pmatrix}.$$

In bra and ket notation, the left and right circular polarization states are

$$|L\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle), \quad |R\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle),$$

$$|H\rangle = \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle), \quad |V\rangle = \frac{i}{\sqrt{2}}(|R\rangle - |L\rangle).$$

Now we need to find matrix elements:

$$\langle H|\hat{\gamma}_C|H\rangle = \left(\frac{1}{\sqrt{2}}(|L\rangle + |R\rangle)\left(|L\rangle\langle L| - |R\rangle\langle R|\right)\left(\frac{1}{\sqrt{2}}(|L\rangle + |R\rangle)\right)\right)$$

$$= \frac{1}{2} \left(|L\rangle + |R\rangle\right)\left(|L\rangle - |R\rangle\right) = 0.$$
\[ \langle H|\hat{\phi}_C|V\rangle = \left( \frac{1}{\sqrt{2}}(\langle L| + \langle R|) \right) \left( |L\rangle\langle L| - |R\rangle\langle R| \right) \left( \frac{i}{\sqrt{2}}(|R\rangle - |L\rangle) \right) \]
\[ = \left( \frac{1}{\sqrt{2}}(\langle L| + \langle R|) \right) \left( \frac{-i}{\sqrt{2}}(|L\rangle + |R\rangle) \right) \]
\[ = -\frac{i}{2}(2) = -i. \]
\[ \langle V|\hat{\phi}_C|H\rangle = \left( \frac{-i}{\sqrt{2}}(\langle R| - \langle L|) \right) \left( |L\rangle\langle L| - |R\rangle\langle R| \right) \left( \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle) \right) \]
\[ = \left( \frac{-i}{\sqrt{2}}(\langle R| - \langle L|) \right) \left( \frac{1}{\sqrt{2}}(|L\rangle - |R\rangle) \right) \]
\[ = -\frac{i}{2}(-2) = i. \]
\[ \langle V|\hat{\phi}_C|V\rangle = \left( \frac{-i}{\sqrt{2}}(\langle R| - \langle L|) \right) \left( |L\rangle\langle L| - |R\rangle\langle R| \right) \left( \frac{i}{\sqrt{2}}(|R\rangle - |L\rangle) \right) \]
\[ = \left( \frac{-i}{\sqrt{2}}(\langle R| - \langle L|) \right) \left( \frac{i}{\sqrt{2}}(|L\rangle + |R\rangle) \right) = 0. \]

So the matrix representation of \( \hat{\phi}_C \) can be expressed as
\[
\hat{\phi}_C = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\]

Note that is \( \hat{\phi}_C \) Hermitian, \( \hat{\phi}_C^\dagger = \hat{\phi}_C \).

(b) For the first measurement along the \( \{ |H\rangle, |V\rangle \} \) basis yielding +1, the required probability is
\[
P(+1|L\rangle) = |\langle H|L\rangle|^2
\]
Now \( \langle H|L\rangle = \langle H\rangle \left( \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle) \right) = \frac{1}{\sqrt{2}}. \)
So \( P(+1|L\rangle) = \left( \frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2}. \)

For the second measurement of the polarization in the circular \( \{ |L\rangle, |R\rangle \} \) basis, the probability of obtaining -1 is
\[
P(-1|L\rangle) = |\langle R|L\rangle|^2
\]
\[= \langle L|R\rangle \langle R|L\rangle = 0. \]

3. Verify that measurements of \( \varphi_{HV} \) and \( \varphi_{45} \) satisfy the appropriate indeterminacy relation for a beam of photons prepared in the state
\[
|\psi\rangle = \frac{1}{\sqrt{3}}|H\rangle + \sqrt{\frac{2}{3}}e^{i\pi/3}|V\rangle.
\]
Answer

The indeterminacy relation that we need to verify is
\[
\Delta \varphi_{HV} \Delta \varphi_{45} \geq \frac{1}{2}|\langle [\hat{\varphi}_{HV}, \hat{\varphi}_{45}] \rangle|.
\]
(1)

We have to find \(\Delta \varphi_{HV}\) and \(\Delta \varphi_{45}\) for a beam of photons prepared in the state \(|\psi\rangle\).

Evaluate \(\Delta \varphi_{HV}\) by finding
\[
\langle \hat{\varphi}_{HV} \rangle = \langle \psi | \hat{\varphi}_{HV} | \psi \rangle,
\]
where \(\hat{\varphi}_{HV} = (+1)|H\rangle\langle H| + (-1)|V\rangle\langle V|\), so we have
\[
\langle \hat{\varphi}_{HV} \rangle = \left( \frac{1}{\sqrt{3}}|H\rangle + \sqrt{\frac{2}{3}} e^{-i\pi/3} |V\rangle \right) \left( |H\rangle\langle H| - |V\rangle\langle V| \right) \left( \frac{1}{\sqrt{3}}|H\rangle + \sqrt{\frac{2}{3}} e^{i\pi/3} |V\rangle \right)
\]
\[
= \frac{1}{3} \left( |H\rangle + \sqrt{2} e^{-i\pi/3} |V\rangle \right) \left( |H\rangle - \sqrt{2} e^{i\pi/3} |V\rangle \right)
\]
\[
= \frac{1}{3} (1 - 2) = -\frac{1}{3}.
\]

As we solved in class that \(\hat{\varphi}^2_{HV} = \hat{1}\), so the variance \(\Delta \varphi^2_{HV}\) becomes
\[
\Delta \varphi^2_{HV} = \langle \hat{\varphi}^2_{HV} \rangle - \langle \hat{\varphi}_{HV} \rangle^2
\]
\[
= \langle \psi | \hat{1} | \psi \rangle - \left( -\frac{1}{3} \right)^2
\]
\[
= 1 - \frac{1}{9} = \frac{8}{9},
\]
and the uncertainty is \(\Delta \varphi_{HV} = \sqrt{\Delta \varphi^2_{HV}} = 2\sqrt{2}/3\).

Similarly, we find \(\Delta \varphi_{45}\) by finding \(\langle \hat{\varphi}_{45} \rangle\) and \(\langle \hat{\varphi}^2_{45} \rangle\):
\[
\langle \hat{\varphi}_{45} \rangle = \langle \psi | \hat{\varphi}_{45} | \psi \rangle,
\]
where \(\hat{\varphi}_{45}\) can be expressed as
\[
\hat{\varphi}_{45} = (+1)|45^\circ\rangle\langle 45^\circ| + (-1)|-45^\circ\rangle\langle -45^\circ|.
\]
\[
= \left( \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) \right) \left( \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle) \right) - \left( \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle) \right) \left( \frac{1}{\sqrt{2}}(|H\rangle - |V\rangle) \right)
\]
\[
= |H\rangle\langle V| + |V\rangle\langle H|,
\]
so we have
\[
\langle \hat{\varphi}_{45} \rangle = \left( \frac{1}{\sqrt{3}}|H\rangle + \sqrt{\frac{2}{3}} e^{-i\pi/3} |V\rangle \right) \left( |H\rangle\langle V| + |V\rangle\langle H| \right) \left( \frac{1}{\sqrt{3}}|H\rangle + \sqrt{\frac{2}{3}} e^{i\pi/3} |V\rangle \right)
\]
\[
\psi = \frac{1}{3} \left( \langle H \rangle + \sqrt{2} e^{-i\pi/3} \langle V \rangle \right) \left( \langle H \rangle \sqrt{2} e^{i\pi/3} + |V\rangle \right)
\]
\[
= \frac{1}{3} \left( \sqrt{2} e^{i\pi/3} + \sqrt{2} e^{-i\pi/3} \right)
\]
\[
= \frac{2\sqrt{2}}{3} \cos(\pi/3) = \frac{\sqrt{2}}{3}.
\]

The variance \( \Delta \varphi_{45}^2 \) is then
\[
\Delta \varphi_{45}^2 = \langle \varphi_{45}^2 \rangle - \langle \varphi_{45} \rangle^2
\]
\[
= \langle \psi | 1 | \psi \rangle - \left( \frac{\sqrt{2}}{3} \right)^2
\]
\[
= 1 - \frac{2}{9} = \frac{7}{9},
\]
\[
\Delta \varphi_{45} = \sqrt{\Delta \varphi_{45}^2} = \frac{\sqrt{7}}{3}.
\]

Now we have to find commutator of \( \hat{\varphi}_{HV} \) and \( \hat{\varphi}_{45} \),
\[
[\hat{\varphi}_{HV}, \hat{\varphi}_{45}] \equiv \langle \hat{\varphi}_{HV} \hat{\varphi}_{45} - \hat{\varphi}_{45} \hat{\varphi}_{HV} \rangle
\]
\[
= \left( \langle H \rangle \langle H \rangle - |V\rangle \langle V \rangle \right) \left( |H\rangle \langle V \rangle + |V\rangle \langle H \rangle \right) - \left( \langle H \rangle \langle V \rangle + |V\rangle \langle H \rangle \right) \left( \langle H \rangle \langle H \rangle - |V\rangle \langle V \rangle \right)
\]
\[
= |H\rangle \langle V \rangle - |V\rangle \langle H \rangle + |H\rangle \langle V \rangle - |V\rangle \langle H \rangle
\]
\[
= 2(|H\rangle \langle V \rangle - |V\rangle \langle H \rangle).
\]

The expectation value of the commutator is
\[
\langle [\hat{\varphi}_{HV}, \hat{\varphi}_{45}] \rangle = 2 \langle \psi | \left( |H\rangle \langle V \rangle - |V\rangle \langle H \rangle \right) | \psi \rangle
\]
\[
= 2 \left( \frac{1}{\sqrt{3}} \langle H \rangle + \sqrt{\frac{2}{3}} e^{-i\pi/3} \langle V \rangle \right) \left( |H\rangle \langle V \rangle - |V\rangle \langle H \rangle \right) \left( \frac{1}{\sqrt{3}} |H\rangle + \sqrt{\frac{2}{3}} e^{i\pi/3} |V\rangle \right)
\]
\[
= \frac{2}{3} \left( \langle H \rangle + \sqrt{2} e^{-i\pi/3} \langle V \rangle \right) \left( \sqrt{2} e^{i\pi/3} |H\rangle - |V\rangle \right)
\]
\[
= \frac{2\sqrt{2}}{3} \left( e^{i\pi/3} - e^{-i\pi/3} \right)
\]
\[
= \frac{i4\sqrt{2}}{3} \sin(\pi/3) = 2i \sqrt{\frac{2}{3}}.
\]

Substituting values into Eq(1) yields
\[
\left( \frac{2\sqrt{2}}{3} \right) \left( \sqrt{\frac{7}{3}} \right) \geq \frac{1}{2} \left( 2i \sqrt{\frac{2}{3}} \right)
\]
\[
\frac{2\sqrt{7}}{9} \geq \frac{1}{\sqrt{3}} \\
1.76 \geq 1.73,
\]

which shows that the indeterminacy relation is satisfied.

4. If \( \hat{A} \) and \( \hat{B} \) are Hermitian, and \([\hat{A}, \hat{B}] = i\hat{C}\), prove that \( \hat{C} \) is Hermitian. Given this definition of \( \hat{C} \), you will often see the indeterminacy principle written as

\[
\Delta A \Delta B \geq \frac{1}{2}|\langle \hat{C} \rangle|.
\]

**Answer**

The commutator of \( \hat{A} \) and \( \hat{B} \) is defined as

\[
[\hat{A}, \hat{B}] \equiv (\hat{A}\hat{B} - \hat{B}\hat{A}) \\
i\hat{C} = (\hat{A}\hat{B} - \hat{B}\hat{A}) \\
\hat{C} = -i(\hat{A}\hat{B} - \hat{B}\hat{A}),
\]

then

\[
\hat{C}^\dagger = i(\hat{A}\hat{B} - \hat{B}\hat{A})^\dagger \\
= i(\hat{B}^\dagger \hat{A}^\dagger - \hat{A}^\dagger \hat{B}^\dagger) \\
= i(\hat{B} \hat{A} - \hat{A} \hat{B}) \quad \because \hat{A} \text{ and } \hat{B} \text{ are Hermitian} \\
= -i(\hat{A}\hat{B} - \hat{B}\hat{A}) = \hat{C},
\]

so \( \hat{C} \) is Hermitian. Using the indeterminacy principle derived in class,

\[
\Delta A \Delta B \geq \left|\frac{1}{2i}\left[\hat{A}, \hat{B}\right]\right|,
\]

and for \([\hat{A}, \hat{B}] = i\hat{C}\) indeterminacy principle is written as

\[
\Delta A \Delta B \geq \frac{1}{2}|\langle \hat{C} \rangle|.
\]

5. Prove the Schwartz inequality: \( \langle a | a \rangle \langle b | b \rangle \geq |\langle a | b \rangle|^2 \). Hint: Let

\[
|\psi\rangle = |a\rangle - \left(\frac{\langle b | a \rangle}{\langle b | b \rangle}\right)|b\rangle,
\]

Due date: Feb. 28, 2014, 5:00pm
Answer

For the given state $j\rangle$, 

$$ |\psi\rangle = |a\rangle - \left( \frac{\langle b|a \rangle}{\langle b|b \rangle} \right) |b\rangle, $$

we know that 

$$ \langle \psi | = \langle a | - \left( \frac{\langle a|b \rangle}{\langle b|b \rangle} \right) \langle b |. $$

Using $\langle \psi |\psi \rangle \geq 0$, we have 

$$ \langle \psi |\psi \rangle = \left( \langle a | - \left( \frac{\langle a|b \rangle}{\langle b|b \rangle} \right) \langle b | \right) \left( \langle a | - \left( \frac{\langle a|b \rangle}{\langle b|b \rangle} \right) \langle b | \right) $$

$$ = \langle a|a \rangle - \frac{\langle b|a \rangle}{\langle b|b \rangle} \langle a|b \rangle - \frac{\langle a|b \rangle}{\langle b|b \rangle} \langle b|a \rangle + \frac{\langle a|b \rangle}{\langle b|b \rangle} \langle b|a \rangle \frac{\langle b|a \rangle}{\langle b|b \rangle} \langle b|b \rangle $$

$$ = \langle a|a \rangle - \frac{\langle a|b \rangle^2}{\langle b|b \rangle} $$

$$ \Rightarrow \langle a|a \rangle \geq \frac{\langle a|b \rangle^2}{\langle b|b \rangle} $$

$$ \langle a|a \rangle \langle b|b \rangle \geq \langle a|b \rangle^2, $$

which is the required result.

6. (a) $\hat{A}$ is a Hermitian operator. Let $\{|1\rangle, |2\rangle, \ldots, |N\rangle\}$ form an orthonormal basis for the Hilbert space containing an arbitrary state $|\psi\rangle$. Show that 

$$ \langle \hat{A}^2 \rangle = \sum_{i=1}^{N} |\langle \psi |\hat{A}|i \rangle|^2. $$

(b) If $\{|\lambda_1\rangle, |\lambda_2\rangle, \ldots, |\lambda_N\rangle\}$ are nondegenerate eigenstates of the Hermitian operator $\hat{A}$ with eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, show that for an arbitrary state $|\psi\rangle$, 

$$ \langle \hat{A} \rangle = \sum_{i=1}^{N} \lambda_i |\langle \psi |\lambda_i \rangle|^2. $$
Answer

(a) The expectation value of $\hat{A}^2$ can be found as

$$\langle \hat{A}^2 \rangle = \langle \psi | \hat{A} \hat{A} | \psi \rangle$$

$$= \langle \psi | \hat{A} \hat{1} \hat{A} | \psi \rangle$$

$$= \langle \psi | \hat{A} \left( \sum_i |i\rangle \langle i| \right) \hat{A} | \psi \rangle$$

$$= \sum_i \langle \psi | \hat{A} |i\rangle \langle i| \hat{A} | \psi \rangle$$

$$= \sum_i \langle i| \hat{A}^\dagger |\psi \rangle \langle \psi | \hat{A} |i\rangle \quad \therefore \hat{A} \text{ is Hermitian}$$

$$= \sum_i \langle \psi | \hat{A} |i\rangle |^2,$$

is the required relation.

(b) Similarly, following the same procedure as we did in part (a)

$$\langle \hat{A} \rangle = \langle \psi | \hat{A} | \psi \rangle$$

$$= \langle \psi | \hat{A} \hat{1} | \psi \rangle$$

$$= \langle \psi | \hat{A} \left( \sum_i |\lambda_i\rangle \langle \lambda_i| \right) | \psi \rangle$$

$$= \sum_i \langle \psi | \hat{A} |\lambda_i\rangle \langle \lambda_i| \psi \rangle$$

$$= \sum_i \lambda_i \langle \psi | \lambda_i \rangle \langle \lambda_i| \psi \rangle$$

$$= \sum_{i=1}^{N} \lambda_i \langle \psi | \lambda_i \rangle |^2.$$
