Quiz 6: Modern Physics
Solution

Name: 
Roll no:
Attempt all questions.

Some universal constants:

\[ h = \text{Planck's constant} = 6.63 \times 10^{-34} \text{ Js} \]
\[ \hbar = \text{Reduced Planck's constant} = 1.06 \times 10^{-34} \text{ Js} \]
\[ 1\text{eV} = 1.6 \times 10^{-19} \text{ J} \]

TDSE: \[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \Psi(x, t) + V(x)\Psi(x, t) = i\hbar \frac{d}{dt}\Psi(x, t) \]

TISE: \[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + V(x)\psi(x) = E\psi(x). \]

1. The figure shows the potential energy landscape and the dashed line shows the energy of an electron trapped inside the infinite well.

![Potential Energy Landscape](image)

The possible wave function is:

![Wave Functions](image)
Answer 1:

From the figure we can see that the wavenumber for region \( 0 \leq x \leq L/2 \) is,

\[
    k = \frac{\sqrt{2mE}}{\hbar},
\]

while the wavenumber for region \( L/2 \leq x \leq L \) is,

\[
    k' = \frac{\sqrt{2m(E - V_0)}}{\hbar}.
\]

Since \( k' < k \), \( \lambda > \lambda' \), option (d) is the correct answer. The wavelength is longer in the right half of the potential well.

2. A particle of mass \( m \) is trapped inside an infinite well of length \( L \). The wavefunction at time \( t = 0 \) is,

\[
    \Psi(x, 0) = \sqrt{\frac{3}{5L}} \sin \left( \frac{3\pi x}{L} \right) + \sqrt{\frac{2}{5L}} \sin \left( \frac{5\pi x}{L} \right).
\]

One measures the energy of the particle. What does the experimenter record? The ground state energy is \( E_0 \).

(a) The experimenter always measure the energy \( 9E_0 \).

(b) The experimenter will measure the average energy \( \frac{9E_0 + 25E_0}{2} \).

(c) The experimenter measures \( 9E_0 \) with probability \( \frac{1}{2} \) and \( 25E_0 \) with probability \( \frac{1}{2} \).

(d) The experimenter measures \( 9E_0 \) with probability \( \sqrt{\frac{3}{5L}} \) and \( 25E_0 \) with probability \( \sqrt{\frac{2}{5L}} \).

(e) The experimenter measures \( 9E_0 \) with probability \( \frac{3}{5L} \) and \( 25E_0 \) with probability \( \frac{2}{5L} \).

Answer 2:

The correct option is (e).

3.
A particle (photon) is injected from the left into the region I, it encounters a potential step of height \( V_0 \) and enters region II. The energy of the photon is \( 2V_0 \). What can you say about the ratio of the propagation speeds of the photon in regions I and II, \( v_I \) and \( v_{II} \)?

(a) \( v_I = v_{II} \).
(b) \( v_I < v_{II} \).
(c) \( v_I = \frac{v_{II}}{\sqrt{2}} \).
(d) \( v_I = \sqrt{2}v_{II} \).
(e) insufficient information is available to answer this question.

**Answer 3:**

Option (d) is the correct answer. From the figure, we observe that in region I the wavefunction is,

\[
k_I^2 = \frac{2mE}{\hbar^2} = \frac{2m(2V_0)}{\hbar^2} = \frac{4mV_0}{\hbar^2},
\]

and likewise for region II,

\[
k_{II}^2 = \frac{2mE}{\hbar^2} = \frac{2m(V_0)}{\hbar^2} = \frac{2mV_0}{\hbar^2}.
\]

Hence,

\[
\frac{k_I}{k_{II}} = \sqrt{2}.
\]

Since \( k = p/\hbar = mv/\hbar \), we have

\[
v_I = \sqrt{2}v_{II}.
\]
A continuous laser, which emits monochromatic light of frequency $\omega_0$, shines on a shutter. The shutter opens for a small duration of time $\tau$. The light then falls on a spectrometer that measures the intensity versus frequency $\omega$. The spread of the peak is $\Delta\omega$ and is centered at $\omega_0$. If $\tau$ is increased,

(a) $\omega_0$ and $\Delta\omega$ remain unchanged.

(b) both $\omega_0$ and $\Delta\omega$ increase.

(c) both $\omega_0$ and $\Delta\omega$ decrease.

(d) $\omega_0$ remains unchanged while $\Delta\omega$ increases.

(e) $\omega_0$ remains unchanged while $\Delta\omega$ decreases. [3 Marks]

Answer 4:

Option (e) is the correct answer. As $\tau$ increases, by energy-time uncertainty principle

$$\Delta E \Delta t \geq \frac{\hbar}{2} \Rightarrow \Delta\omega \geq \frac{1}{2\Delta t} = \frac{1}{\tau}, \Delta\omega \text{ decreases.}$$

Opening the shutter for longer duration $\tau$ means that the spectrometer is observing the impingent light for longer times, hence the spread in frequency decreases. There is no reason why $\omega_0$ should change, which is an average energy, and not a spread in frequencies.

5. A free electron of energy $E$ has a de Broglie wavelength $\lambda = h/p = h/\sqrt{2mE}$ and speed $v$. In the presence of an electric field, it acquires a potential energy $-eu(x)$, where $u(x)$ is the potential. Hence the total energy changes, and the speed of the electron changes to $v'$. What is the value of refractive index $n = \frac{v}{v'}$?

(a) 1 (one).

(b) $\sqrt{E/u(x)}$.

(c) $\sqrt{E/(E - eu(x))}$.

(d) $\sqrt{(E - eu(x))/E}$.

(e) $\sqrt{eu(x)/E}$. [3 Marks]

Answer 5:

In the absence of electric field,

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}, \text{ and speed is } v.$$
In the presence of electric field, total energy of the electron changes to \((E - eu(x))\), resulting in the modified wavelength,

\[ \lambda' = \frac{h}{p'} = \frac{h}{\sqrt{2m(E - eu(x))}}, \]

and the changed speed is \(v'\).

Ratio between the momentums and speed is,

\[ \frac{p}{p'} = \frac{mv}{mv'} = \frac{v}{v'} = \sqrt{\frac{2mE}{2m(E - eu(x))}} = \sqrt{\frac{E}{(E - eu(x))}}. \]

Hence (c) is the correct answer.

6. A particle is described by the wavefunction \(\Psi(x, t) = e^{i(kx - \omega t)}\) and can be thought of a plane wave traveling along the \(x\) axis. The real part at \(t = 0\) is shown in the accompanying diagram. (The wavefunction extends from \(-\infty \) to \(\infty\) which of course we cannot show on paper.) Which of the following statements most accurately describes the probability of finding the particle.

(a) It is equally likely to find the particle anywhere along the \(x\) axis.
(b) It is most likely to be found in the peaks of the wave.
(c) It is most likely to be found in the peaks or the troughs the wave.
(d) The position of the particle depends on \(when\) I make a measurement.
(e) I have no idea how to answer this question. [3 Marks]
**Answer 6:**

The correct option is (a). The probability density $\Psi^*(x,t)\Psi(x,t) = e^{-i(kx-\omega t)}e^{i(kx-\omega t)} = 1$ is independent of position. So the particle can be found, with equal likelihood, anywhere along the $x$ axis.

7. An electron is trapped in a quantum dot of diameter $L$. The electron is in a potential well of depth $V_0$.

![Diagram of a quantum dot with potential well](image)

The energy values are approximately the same as the infinite well, $E = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$. A laser photon of energy $E_{\text{photon}}$ shines on the quantum dot in the ground state. What should be the minimum diameter if the electron is to always remain confined in the quantum dot? [4 Marks]

**Answer 7:**

The energy of electron is $\frac{n^2 \pi^2 \hbar^2}{2mL^2}$ and in the ground state $E_1 = \frac{\pi^2 \hbar^2}{2mL^2}$. If a photon shines on the quantum dot, the electron can be excited to the level, $E_{\text{photon}} + \frac{\pi^2 \hbar^2}{2mL^2}$. If the electron is to remain confined,

$$E_{\text{photon}} + \frac{\pi^2 \hbar^2}{2mL^2} < V_0$$

$$\frac{\pi^2 \hbar^2}{2mL^2} < V_0 - E_{\text{photon}}$$

$$\frac{\pi^2 \hbar^2}{2mL^2} > \frac{V_0 - E_{\text{photon}}}{1}$$

$$L > \sqrt{\frac{\pi^2 \hbar^2}{2m(V_0 - E_{\text{photon}})}}$$

which is the minimum diameter that the quantum dot must have.

8. List two ways of experimentally determining the value of Planck's constant ($h$). [4 Marks]
Answer 8:

(i) We can determine $h$ using a photoelectric effect experiment. Vary the incident frequency $f$ and find the stopping potential $V_0$. The slope of the curve is $e/h$ and intercept is $\phi/h$, where $\phi$ is the work function.

(ii) In an XRF (X-ray fluorescence) experiment, measure the energy of the $K_a$ X-ray. Since energy level depends on $h$, $E_n = -\frac{mc^4}{32\pi^2\epsilon_0h^2} \cdot \frac{1}{n^4}$, measuring the wavelength (or energy) can be used to determine $h$ or $h$.

There can be other methods as well to determine $h$.

9. The position wavefunction, $\psi(x)$ of a particle at some instant is given by,

$$\psi(x) = \frac{1}{L^2 + (x-x_0)^2/\alpha^2},$$

where $L, x_0, \alpha$ are constants. Which of the following expressions given below is a good approximate to the spread in the momentum, $\Delta p$? We are measuring spreads by FWHM (full width at half maximum), and $\Delta x \Delta p \geq h$. (HINT: The wavefunction is maximum at $x = x_0$) and its profile is shown in the figure.
(a) $\Delta p \sim \hbar / (\alpha L)$

(b) $\Delta p \sim \hbar / \alpha$

(c) $\Delta p \sim \hbar / (\alpha^2 x_o)$

(d) $\Delta p \sim \hbar / (\alpha^2 L^2)$

(e) $\Delta p \sim \hbar / (\alpha x_o)$

[3 Marks]

**Answer 9:**

We are given that,

$$\psi(x) = \frac{1}{L^2 + (x - x_0)^2 / \alpha^2}.$$  

At $x = x_0$,

$$\psi(x = x_0) = \frac{1}{L^2}.$$  

We can calculate $(x - x_0)$ where the amplitude is half of maximum, i.e., $\psi(x) = \frac{1}{2L^2}$.

$$\Rightarrow \frac{1}{2L^2} = \frac{1}{L^2 + (x - x_0)^2 / \alpha^2}$$

$$2L^2 = L^2 + \frac{(x - x_0)^2}{\alpha^2}$$

$$L^2 = \frac{(x - x_0)^2}{\alpha^2}$$

$$\alpha^2 L^2 = (x - x_0)^2$$

$$\alpha L = (x - x_0)$$

$$\Delta x \sim 2\alpha L.$$  

According to uncertainty principle,

$$\Delta p \geq \frac{\hbar}{\Delta x} = \frac{\hbar}{2\alpha L}.$$  

Since we are talking about estimates, option (a) is the correct answer.

10. Hydrogen’s 656 nm red spectral line is the result of a transition between quantum states of the electron in the hydrogen atom. Such transitions occur within approximately $10^{-8}$ s. Using the uncertainty principle $\Delta E \Delta t \geq \hbar / 2$, find the range of wavelengths observed.

[3 Marks]
Answer 10:

We are given that $\lambda = 656 \text{ nm}$ and $\Delta t = 10^{-8} \text{ s}$. From the uncertainty principle,

$$\Delta E \Delta t \geq \frac{h}{2}.$$  \hspace{1cm} (1)

Since $E = \frac{hc}{\lambda}$ we have,

$$\Delta E = -\frac{hc \Delta \lambda}{\lambda^2}.$$  

Inserting these values in equation (1),

$$\left| -\frac{hc \Delta \lambda}{\lambda^2} \Delta t \right| \geq \frac{h}{2}$$

$$\Delta \lambda \geq \frac{h}{2hc\Delta t} \frac{\lambda^2}{\lambda^2}$$

$$= \frac{h}{4\pi c \Delta t}$$

$$= \frac{(656 \times 10^{-9} \text{ m})^2}{4\pi \times 3 \times 10^8 \text{ m/s} \times 10^{-8} \text{ s}}$$

$$= 1.1 \times 10^{-14} \text{ m},$$

which is the range of wavelengths.