Recitation: Thanks Mr. Planck that your constant is small!

Solution

1. A stationary 1 mg grain of sand is found to be at a given location within an uncertainty of 550 nm.

(a) What is the minimum uncertainty in its velocity?

(b) Were it moving at this speed, how long would it take to travel 1 µm?

(c) Can classical mechanics be applied reliably?

(d) What is a reasonable wavelength of the grain of sand and will it behave as a wave or as a particle?

(e) What is the minimum uncertainty in its velocity if $h = 6.67 \times 10^{-10}$ Js instead of $6.67 \times 10^{-34}$ Js.

Answer 1: We are given that,

Mass of grain = $m = 1 \text{ mg} = 10^{-6}$ kg

Uncertainty in position = $\Delta x = 550 \text{ nm} = 550 \times 10^{-9}$ m.

(a) Uncertainty in velocity can be calculated by calculating uncertainty in its momentum. According to uncertainty principle, the minimum uncertainty is approximately,

$$\Delta x \Delta p \geq \frac{h}{2}$$

$$\Rightarrow \Delta p \geq \frac{h}{2\Delta x} = \frac{1.05 \times 10^{-34}}{2 \times 550 \times 10^{-9} \text{ m}}$$

$$= 9.65 \times 10^{-29} \text{ kg m s}^{-1}.$$  

$\Delta p$ is small because $h$ is small. Now the uncertainty in speed is calculated as,

$$\Delta p = m\Delta v$$

$$\Delta v = \frac{\Delta p}{m}$$

$$= \frac{9.7 \times 10^{-29} \text{ kg m s}^{-1}}{10^{-6} \text{ kg}}$$

$$= 9.65 \times 10^{-23} \text{ m s}^{-1}.$$
For macroscopic particle \( \Delta v \geq \frac{\hbar}{2(\Delta x)m} \) is small because of the very small \( \hbar/m \) ratio. \( \Delta v \) becomes significant only if \( \hbar \) were large or the mass \( m \) decreases. Small \( \hbar \) and large \( m \) makes the macroscopic classic world “undisturbed” by quantum uncertainties!

(b) 

\[
\Delta t \approx \frac{1 \mu m}{\Delta v} = \frac{10^{-6}}{9.65 \times 10^{-23}} s = 0.1 \times 10^{17} s \approx 3 \text{ billion years!}
\]

The uncertainty in velocity is really really small! An observer would require 3 billion years to notice the grain of sand, supposedly at rest, at a position 1 \( \mu m \) away from its original position. The current age of the solar system is approximately 5 billion years.

(c) Yes uncertainties are extremely small. No device has ever been built, and may never be built that can measure these small velocities. We can safely apply classical mechanics to a grain of sand; there is effectively no uncertainty in position or in momentum. Furthermore, a precision as fine as \( 10^{-22} \) m/s is never required in classical mechanics.

(d) 

\[
\lambda = \frac{h}{mv} = \frac{h}{p}.
\]

Now what momentum should I choose? The uncertainty principle dictates a \( \Delta p \sim 9.7 \times 10^{-29} \) kg m s\(^{-1}\). The momentum could therefore have any value between, approximately \( -\Delta p/2 \) and \( \Delta p/2 \). Let’s choose an extreme value, \( p \sim \Delta p/2 \sim 5 \times 10^{-29} \) kg m s\(^{-1}\). Therefore, 

\[
\lambda \sim \frac{6.67 \times 10^{-34}}{5 \times 10^{-29}} \sim 1.3 \times 10^{-5} \text{ m.}
\]

This is such a small wavelength compared to apparatus we might use for macroscopic objects, that for all practical purposes, the grain of sand acts like a particle!

(e) \( \Delta v \) would be \( 9.65 \times 10 \approx 96 \) m/s, if \( \hbar \) were this large. This is a huge uncertainty. We are “saved” by the exceedingly small value of \( \hbar \)

2. An electron is held in orbit about a proton by electrostatic attraction.

(a) Assume that an “orbiting electron wave” has the same energy an orbiting particle would have if at radius \( r \) and of momentum \( mv \). Write an expression for this energy.
(b) If the electron behaves as a classical particle, it must obey $F = ma$. Assuming circular orbit, apply $F = ma$ to eliminate $v$ in favor of $r$ in the energy expression.

(c) Suppose instead that the electron is an orbiting wave, and that the product of the uncertainties in radius $r$ and momentum $p$ is governed by an uncertainty relation of the form $\Delta p \Delta r \approx \hbar$. Also assume that a typical radius of this orbiting wave is roughly equal to the uncertainty $\Delta r$, and that a typical magnitude of the momentum is roughly equal to the uncertainty $\Delta p$, so that the uncertainty relation becomes $pr \approx \hbar$. Use this to eliminate $v$ in favor of $r$ in the energy expression.

(d) Sketch on the same graph the expressions from parts (b) and (c).

(e) Find the minimum possible energy for the orbiting electron wave, and the value of $r$ to which it corresponds.

**Answer 2:**

(a) Total energy of an orbiting particle in terms of its kinetic energy and electrostatic potential energy is give by,

$$E_{\text{total}} = K.E. + P.E.$$  
$$E_{\text{total}} = \frac{1}{2}mv^2 + \frac{k\varepsilon^2}{r}$$  
$$E_{\text{total}} = \frac{1}{2}mv^2 - \frac{k\varepsilon^2}{r},$$

where $k = 1/4\pi\varepsilon_0$. Therefore total energy of the particle will become,

$$E_{\text{total}} = \frac{1}{2}mv^2 - \frac{\varepsilon^2}{4\pi\varepsilon_0 r}.$$  

Hence the energy of “orbiting electron wave” is also,

$$E = \frac{1}{2}mv^2 - \frac{\varepsilon^2}{4\pi\varepsilon_0 r}.$$  

(1)

(b) Since electron is orbiting in a circular orbit, its centripetal acceleration in its orbit is $(v^2/r)$, while electrostatic force on the electron is $(ke^2/r^2)$, thus,

$$F = ma$$

$$\frac{ke^2}{r^2} = m\left(\frac{v^2}{r}\right)$$

$$\Rightarrow v^2 = \frac{ke^2}{mr}$$

$$v^2 = \frac{\varepsilon^2}{4\pi\varepsilon_0 mr}.$$
Use this value of $v^2$ in equation (1),

$$E_{\text{classical particle}} = \frac{1}{2} m \left( \frac{e^2}{4\pi\varepsilon_0 mr} \right) - \frac{e^2}{4\pi\varepsilon_0 r}$$

$$= \frac{1}{2} \left( \frac{e^2}{4\pi\varepsilon_0 r} \right) - \frac{e^2}{4\pi\varepsilon_0 r}$$

$$= \frac{e^2}{8\pi\varepsilon_0 r} - \frac{e^2}{4\pi\varepsilon_0 r}$$

$$= \frac{e^2}{8\pi\varepsilon_0 r}.$$  

The negative electrostatic potential energy is always of greater magnitude than the positive kinetic energy, so the total energy strictly decreases as $r$ decreases. Hence there is no minimum energy. In the accompanying figure, course $A$ corresponds to the energy of the classical particle, whose energy decreases as $r$.

(c) Now assuming $pr = \hbar$, we have $p = \hbar/r$ or $v = \hbar/mr$. Therefore equation (1) becomes,

$$E_{\text{matter wave}} = \frac{1}{2} m \left( \frac{\hbar}{mr} \right)^2 - \frac{e^2}{4\pi\varepsilon_0 r}$$

$$= \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\varepsilon_0 r}.$$  

In this case as $r$ decreases, and the wave become more compact, the likely speed increases. The kinetic energy increases faster than the potential decreases, and the total energy at some point must increase. Hence applying uncertainty principle there is a turning point $A$ in the curve labelled $B$.

(d) The two plots are shown in the figure.

While the energy of a classical particle would monotonically decrease as $r$ decreases, the energy of the matter wave reaches a minimum, and then increases.
(e) The minimum possible energy for the orbiting electron wave can be calculated by setting the derivative of energy with respect to $r$, to zero.

\[
E_{\text{matter wave}} = \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\varepsilon_0 r}
\]

\[
\frac{dE_{\text{matter wave}}}{dr} = -\frac{\hbar^2}{mr^3} + \frac{e^2}{4\pi\varepsilon_0 r^2} = 0
\]

\[
\Rightarrow \quad r = \frac{4\pi\varepsilon_0 \hbar^2}{me^2}
\]

\[
= \frac{9 \times 10^9 \text{ Nm}^2/\text{C}^2 \times (1.055 \times 10^{-34} \text{ Js})^2}{9.11 \times 10^{-31} \times (1.6 \times 10^{-19} \text{ C})^2}
\]

\[
= 5.3 \times 10^{-11} \text{ m.}
\]

This turns out to be astoundingly close to the Bohr radius calculated earlier in class. Inserting this value of $r$ and other constants will give energy for matter wave as follows.

\[
E_{\text{matter wave}} = -13.6 \text{ eV.}
\]

The energy happens to equal the correct, experimentally determined value, and the radius is indeed the most probable radius at which the electron would be found. That these agree so closely is an accident; many approximations have been made. Nevertheless, the uncertainty principle does impose a lower limit on the energy, and it is no accident that the value we obtained is of the correct order of magnitude.