1. Consider a system of two Einstein solids, $A$ and $B$, each containing 10 oscillators, sharing a total of 20 units of energy. Assume that the solids are brought close and interact, and that the total energy is fixed.

(a) How many different macrostates are available to this system?

(b) How many different microstates are available to this system?

(c) Assuming that this system is in thermal equilibrium, what is the probability of finding all the energy in solid $A$?

(d) What is the probability of finding exactly half of the energy in solid $A$?

(e) Under what circumstances would this system exhibit irreversible behavior?

Answer 1:

(a) We are given that,

No. of oscillators in solid $A = N_A = 10$

No. of oscillators in solid $B = N_B = 10$

No. of quantums in solid $A = q_A = 10$

No. of quantums in solid $B = q_B = 10$.

There are eleven macrostates when considering the distribution of the 10 quantums between the two atoms.

(b) However the number of microstates that can be calculated by the equation of multiplicity of Einstein solid (interacting solids), $\Omega = \frac{(q+N-1)!}{q!(N-1)!}$ is much larger as seen in the Table below.

<table>
<thead>
<tr>
<th>$q_A$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_B$</td>
<td>10</td>
<td>9</td>
<td>8</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\Omega_A$</td>
<td>1</td>
<td>10</td>
<td>55</td>
<td>220</td>
<td>715</td>
<td>2002</td>
<td>5005</td>
<td>11440</td>
<td>24310</td>
<td>48620</td>
<td>92378</td>
</tr>
<tr>
<td>$\Omega_B$</td>
<td>92378</td>
<td>48620</td>
<td>24310</td>
<td>11440</td>
<td>5005</td>
<td>2002</td>
<td>715</td>
<td>220</td>
<td>55</td>
<td>10</td>
<td>1</td>
</tr>
<tr>
<td>$\Omega_A\Omega_B$</td>
<td>92378</td>
<td>486200</td>
<td>1337050</td>
<td>2516800</td>
<td>3578575</td>
<td>4008004</td>
<td>3578578</td>
<td>2516800</td>
<td>1337050</td>
<td>486200</td>
<td>92378</td>
</tr>
</tbody>
</table>

The total number of microstates is the sum of entries in the last row of the table.
(c) If the system is in thermal equilibrium, then the probability of finding all the energy in solid \( A \) is,

\[
\text{probability} = \frac{\text{No. of microstates in solid } A}{\text{Total no. of microstates in the system}} = \frac{92378}{20030010} = 4.6 \times 10^{-3} = 4.6 \times 10^{-1}\% = 0.46\%.
\]

(d) The probability of finding exactly half of the energy in solid \( A \) is,

\[
\text{probability} = \frac{4008004}{20030010} = 0.2 = 20\%.
\]

(e) The system will exhibit nearly irreversible behavior when it starts off from one of the smaller probability macrostates, i.e. when initially, all the energy is stored in either of the objects.

2. Suppose you flip 20 fair coins.

(a) How many possible outcomes (microstates) are there?

(b) What is the probability of getting the sequence HTHHTTHTHHTHHHHTHHTT (in exactly that order)?

(c) What is the probability of getting 12 heads and 8 tails (in any order)?

**Answer 2:**

(a) For the case of “binary” coins, microstates can be calculated by the equation,

\[
\Omega(N, n) = \frac{N!}{q!(N-q)!}
\]

\[
\Rightarrow \Omega(20, 0) = \frac{20!}{0!(20-0)!} = \frac{20!}{0!(20)!} = 1
\]

\[
\Omega(20, 1) = \frac{20!}{1!(20-1)!} = \frac{20!}{1!(19)!} = 20
\]

\[
\Omega(20, 2) = \frac{20!}{2!(20-2)!} = \frac{20!}{2!(18)!} = 190,
\]

and so on.

But there is also a simple formula,

\[
\sum_{n=0}^{N} \Omega(N, n) = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} = 2^n.
\]

We can prove this formula using the binomial theorem,

\[
(x + y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \binom{n}{2} x^{n-2} y^2 + \ldots + \binom{n}{n} x^0 y^n.
\]
Setting \( x = y = 1 \) in the above, we obtain,

\[
(1 + 1)^n = 2^n = \sum_{k=0}^{n} \binom{n}{k} = \frac{n!}{k!(n-k)!}.
\]

Hence

\[
\sum_{n=0}^{20} \Omega(20, n) = 2^{20} = 1,048,576.
\]

(b) Probability of getting the sequence HTHHTTTHTHHTTTHTHTHTHTTHT is, 

\[
\text{Probability} = \frac{1}{1,048,576} = 9.54 \times 10^{-7}.
\]

(c) The number of microstates with exactly 12 heads are,

\[
\Omega(20, 12) = \frac{20!}{12!(20-12)!} = \frac{20!}{12!(8)!} = 9690.
\]

Hence the probability of obtaining 12 heads is,

\[
\text{Probability} = \frac{9690}{1,048,576} = 9.2 \times 10^{-3}.
\]

3. (a) A paramagnet has all its dipoles aligned parallel to a magnet field. What is its entropy?

(b) What is the energy of the system in this state? \( N \) is the total number of dipoles.

(c) If \( N = 10^{23} \), how many microstates are accessible to the system?

(d) If \( N = 10^{23} \), and with the huge number of microstates accessible, can the energy still be zero?

(e) Suppose that the microstates of the system changes a billion times per second. How many microstates will it explore in ten billion years (the age of the universe)?

**Answer 3:**

We are given that, all dipoles in a paramagnet are aligned to the magnetic field, as shown in the figure below.
Hence each dipole has the same energy $-E_{\text{mag}}$. The picture in the energy landscape is shown below,

$$
\begin{array}{ccccccc}
+ & + & + & + & + & + & + \\
- & - & - & - & - & - & - \\
\hline
\text{Total number}=N
\end{array}
$$

Since all the dipoles are necessarily in the ground state, the macrostate corresponding to perfect alignment has $\Omega = 1$. Hence,

$$
S = k_B \ln \Omega = k_B \ln(1) = 0 \text{ J/K}.
$$

(b) If $N$ is the total number of dipoles, then total energy of the system is,

$$
E_{\text{Total}} = -N E_{\text{mag}}.
$$

(c) If $N = 10^{23}$, microstates with $N_\uparrow$ are,

$$
\Omega(N, N_\uparrow) = \Omega(10^{23}, N_\uparrow) = \frac{10^{23}!}{N_\uparrow!10^{23}!},
$$

where $N_\uparrow$ can vary from 1 to $N$.

The total number of microstates accessible,

$$
\sum_{N_\uparrow=0}^{N} \Omega(N, N_\uparrow) = 2^N = 2^{10^{23}},
$$

which is a huge number.

(d) Yes if half of the dipoles are in one direction and the other half in the other, $N_\uparrow = N_\downarrow = N/2$. Total energy is,

$$
E_{\text{Total}} = -\frac{N}{2} E_{\text{mag}} + \frac{N}{2} E_{\text{mag}} = 0.
$$

Total energy is zero but the entropy is maximum.

(d) No. of microstates explored in $10 \times 10^9$ years is,

Microstates explored $\sim 10 \times 10^9$ years $\times 10^9$

$\sim 10 \times 10^9 \times 10^7 \times 10^9$

$\sim 10^{26}$.

Hence even in its lifetime, the system can’t explore all $2^{10^{23}}$ states!