Assignment: Uncertainties and data processing

1. If a ray of light passes from air into glass, then the angles of incidence θ_i and reflection θ_r are related by Snell's law,

$$\sin(\theta_i) = n\sin(\theta_r). \quad (1)$$

The refractive index n can be calculated as,

$$n = \frac{\sin(\theta_i)}{\sin(\theta_r)}. \quad (2)$$

Suppose the angles are measured as,

$$\theta_i = (20 \pm 1)^\circ$$
$$\theta_r = (13 \pm 1)^\circ$$

(a) Find the best estimated value of the refractive index n and uncertainty associated with it.

(b) Calculate the fractional uncertainty in n.

2. A student studying the motion of a cart on air track measures its position, velocity and acceleration at one instant with the results shown in Table (I).

Variable	Best estimated value	Probable range
Position, x	53.3	53.12 to 53.57 (cm)
Velocity, v	-13.5	-14.0 to -13.0 (cm/s)
Acceleration, a	93	96.7 to 90.4 (cm/s^2)

TABLE I: Measurements of position, velocity and acceleration.

Rewrite these results in the standard form $x_{\text{best}} \pm \Delta x$.

3. The Van der Waals equation of state for a non-ideal gas is,

$$\left(P + \frac{a}{V_{\text{m}}^2}\right)(V_{\text{m}} - b) = RT, \quad (3)$$

where P is the pressure, V_m is the molar volume, T is absolute temperature, R is the universal gas constant with a and b species specific for Van der Waals coefficients and given as,

$$\begin{aligned} V_m &= (2.000 \pm 0.003) \times 10^{-4}\,\mathrm{m^3\,mol^{-1}} \\ T &= (298.0 \pm 0.2)\,\mathrm{K} \\ a &= 1.408 \times 10^{-1}\,\mathrm{m^6\,mol^{-2}\,Pa} \\ b &= 3.913 \times 10^{-5}\,\mathrm{m^3\,mol^{-1}} \\ R &= 8.3145\,\mathrm{J\,K^{-1}\,mol^{-1}}. \end{aligned}$$

There is no uncertainty in a, b and R.
Calculate the pressure of the gas and find uncertainty associated with it.

4. Generally, we associate a triangular probability distribution function with analog devices and standard uncertainty can be find out through the second moment of the probability distribution function. Mathematically the triangular probability function $f(x)$ is defined as,

$$\begin{aligned} f(x) &= \frac{(x+a)}{a^2} && \text{for all } x \text{ such that } -a \leq x \leq 0 \\ f(x) &= \frac{(a-x)}{a^2} && \text{for all } x \text{ such that } 0 \leq x \leq a \\ f(x) &= 0 && \text{for all other values of } x \end{aligned}$$

Find the mean (first moment) and variance (second moment) of the above given triangular probability distribution function.

5. To calibrate a prism spectrometer, a student sends light of 10 different wavelengths through the spectrometer and measures the angle θ by which each beam is deflected. For just the first value of λ, he measures θ,

$$52.5,\ 52.3,\ 52.6,\ 52.5,\ 52.7,\ 52.4,\ 52.2,\ 52.5$$

(a) Calculate the mean and standard deviation.

(b) After several measurements, we can expect about 68 % of the observed values to be within σ_m. How many values would you expect to lie within this range ($\bar{\theta} \pm \sigma_m$). How many actually do?

(c) How many values lie within the range of 95 % coverage probability ($\bar{\theta} \pm 2\sigma_{\text{m}}$) and how many lie outside this range?

6. A student measures g using a pendulum of a steel ball suspended by a light string. He records five different lengths of the pendulum and the corresponding time periods T as follows,

Length, l (cm)	51.2	59.7	68.2	79.7	88.3
Time period T (sec)	1.448	1.566	1.669	1.804	1.896

TABLE II: Model table for experimental results.

For each pair, he calculates g through the following expression,

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

(a) Calculate the pair-wise g and find its best estimated value (mean value). Calculate uncertainty in g and quote your final result.

(b) Plot a graph between l and T, find the best estimate of g through the slope of the data given in Table (II).

(c) Which approach do you think yields better result.

7. A student measures the velocity of a glider on a horizontal air track. He uses a multiflash photograph to find the glider's position s at five equally spaced times as shown in Table (III).

Time, t (s)	-4	-2	0	2	4
Position s (cm)	13	25	34	42	56

TABLE III: Experimental data for position and time.

(a) One way to find v would be to calculate ($v = \Delta s/\Delta t$) for each of the four successive two-second intervals and then average them. Show that this procedure gives $v = (s_5 - s_1)/(t_5 - t_1)$, which means that the middle three values are completely ignored by this method. Prove this result.

(b) A better procedure is to make a least squares fit to the equation ($s = s_o + vt$) using all five data points. Follow this procedure to find the best estimate for v and compare your results with that from part (a) (Hint: Use mathematical expressions of least squares fitting of a straight line with equal weights). Find uncertainty in v.

(c) Suppose the time and position is measured through a digital device each with rating of 1 %. Calculated uncertainties in the dependent and independent variables. Plot a graph of the best fit line of the data and show error bars both in the dependent and independent variable.

8. If a steel ball is dropped from a certain height into a container of sand, the impact is called a crater. The relationship between the diameter of the crater and the kinetic energy of the impacting object is given as,

$$D = c\,E^n, \tag{4}$$

where c is a constant, D is the diameter and E is the kinetic energy that can be calculated by assuming that all the kinetic energy possessed by a ball at a height h is converted into potential energy before impact. The data is given in Table (IV).

Mass m (g)	**Height h (cm)**	**Crater diameter D (cm)**
8.4	26	4.0, 4.0, 3.9, 3.9, 4.1, 3.8
28.2	26	5.4, 5.3, 5.0, 5.2, 5.3, 5.1
66.8	26	6.4, 6.4, 6.2, 6.3, 6.2, 6.3
66.8	68	8.2, 7.8, 7.9, 7.9, 8.1, 7.9
66.8	150	10.4, 10.0, 10.1, 10.1, 10.2, 10.2

TABLE IV: Experimental data for crater formation.

(a) Calculate uncertainties in the independent and dependent variables. The mass is measured using a digital weighing balance (rating= 1%), the height and diameter is measured using a ruler (an analog device).

(b) Calculate uncertainties in $\log(E)$ and $\log(D)$. Plot a graph with error bars, both in the dependent and independent variables.

(c) Using transformation rule, transfer all the uncertainties to the dependent variable. Plot a graph with error bars only in the dependent variable.

(d) Calculate the best estimated values of n and c using weighted fit of a straight line. Calculate uncertainties in n and c as well.

Formula sheet:

Slope (m) and intercept (c) with equal weights:

$$m = \frac{\sum_i^N y_i(x_i - \bar{x})}{\sum_i^N (x_i - \bar{x})^2} \quad \text{or} \quad m = \frac{N\sum_i^N x_i y_i - \sum_i^N x_i \sum_i^N y_i}{N\sum_i^N x_i^2 - (\sum_i^N x_i)^2}$$

$$c = \bar{y} - m\bar{x} \quad \text{or} \quad c = \frac{\sum_i^N x_i^2 \sum_i^N y_i - \sum_i^N x_i \sum_i^N x_i y_i}{N\sum_i^N x_i^2 - (\sum_i^N x_i)^2}.$$

Uncertainty in slope m,

$$u_m = \sqrt{\frac{\sum_i^N d_i^2}{D(N-2)}},$$

and,

Uncertainty in intercept c,

$$u_c = \sqrt{\left(\frac{1}{N} + \frac{\bar{x}^2}{D}\right)\left(\frac{\sum_i^N d_i^2}{(N-2)}\right)},$$

where,

$$d_i = y_i - mx_i - c,$$
$$D = \sum_i^N (x_i - \bar{x})^2.$$

Slope m and intercept c with unequal weights

The weights are reciprocal squares of the total uncertainty (u_{Total}),

$$w = \frac{1}{u_{\text{Total}}^2}.$$

The mathematical relationships for slope (m) and intercept (c) are,

$$m = \frac{\Sigma_i w_i \, \Sigma_i w_i (x_i y_i) - \Sigma_i (w_i x_i) \, \Sigma_i (w_i y_i)}{\Sigma_i w_i \, \Sigma_i (w x_i^2) - (\Sigma_i w_i x_i)^2},$$

$$c = \frac{\Sigma_i (w_i x_i^2) \, \Sigma_i (w_i y_i) - \Sigma_i (w_i x_i) \, \Sigma_i (w_i x_i y_i)}{\Sigma_i w_i \, \Sigma_i (w_i x_i^2) - (\Sigma_i w_i x_i)^2},$$

where x is the independent variable, y is the dependent variable and w is the weight. The expressions for the uncertainties in m and c are,

$$u_m = \sqrt{\frac{\Sigma_i w_i}{\Sigma_i w_i \, \Sigma_i (w_i x_i^2) - (\Sigma_i w_i x_i)^2}},$$

$$u_c = \sqrt{\frac{\Sigma_i (w_i x_i^2)}{\Sigma_i w_i \, \Sigma_i (w_i x_i^2) - (\Sigma_i w_i x_i)^2}}.$$