Problems on Wave Mechanics

1. An atom is moving in vacuum with a slow speed \( v \ll c \). A Gaussian wave packet pilots the electron as shown below.

![Wave Packet Diagram]

Show that the group velocity \( v_g \) is equal to the physical speed of the particle. Show that the phase velocity is NOT the speed of the particle. Use the Einstein’s relationship \( E = h\omega \) and the classical energy \( E = \frac{p^2}{2m} \) which is relevant for a slow particle.

2. Now use the relativistic expression for energy \( E = \sqrt{(m_0c^2)^2 + (pc)^2} \) to calculate the phase velocity and group velocity.

This description applies to the question 3 and 4.

As a consequence of the Heisenberg uncertainty principle the more closely an electron is confined to a region of space the higher its kinetic energy will be. In an atom the electrons are confined by the Coulomb potential of the nucleus. The competition between the confining nature of the potential and the liberating tendency of the uncertainty principle gives rise to various quantum mechanical effects. Some of these microscopic effects have repercussions in the way this universe is structured.

3. (a) Use the uncertainty principle to estimate the kinetic energy of an electron confined within a given radius \( r \) in a hydrogen atom. Assume that \( \Delta p \sim p \) and \( \Delta r \sim r \).

(b) Hence estimate the size of the hydrogen atom in its ground state by minimizing it total energy as a function of the orbital radius of the electron.

(c) Compare the size obtained in this way with the value obtained from a Bohr theory calculation.

4. When atoms are subjected to a high enough pressure they become ionized. This will happen, for example, at the center of a sufficiently massive gravitating body.
(a) In order to ionize an atom a certain minimum energy must be supplied to it, 13.6 eV, in the case of hydrogen. Estimate the reduction in atomic radius required to ionize a hydrogen atom.

(b) What pressure $P$ is needed to bring this about? (Hint: $P = -dE/dV$ where $E$ is energy and $V$ is the volume.)

(c) A planet is defined as a body in which the atoms resist the compressive force of gravity. Estimate the maximum mass and size of a planet composed of hydrogen. (You will need to estimate the pressure required at the center of the planet to support a column of mass against its weight.)

This turns out to be of the order of the mass of Jupiter. Thus, Jupiter is not only the largest planet composed of hydrogen in the solar system but anywhere in the universe!

5. The wave function of a free electron is,

$$\psi(x) = Ae^{i(5\times10^{10}x)},$$

where $x$ is in meters. Calculate,

(a) the de Broglie wavelength,

(b) its momentum,

(c) its kinetic energy,

(d) uncertainty in position and momentum.

6. The wave function of a particle confined to move in a one-dimensional box of length $L$ is,

$$\psi(x) = A \sin \left( \frac{2\pi x}{L} \right).$$

(a) Use normalization condition to find the value of $A$.

(b) Calculate the probability of finding the electron between $x = 0$ to $x = L/4$.

7. Consider a particle whose wave function is given by the following expression,

$$\psi(x) = Ae^{-ax^2}.$$

(a) What is the value of $A$ if this wave function is normalized?

(b) What is the expectation value of $x$ for this particle? What does the “expectation value” mean?
8. A stationary 1 mg grain of sand is found to be at a given location within an uncertainty of 550 nm.
   (a) What is the minimum uncertainty in its velocity?
   (b) Were it moving at this speed, how long would it take to travel 1 µm?
   (c) Can classical mechanics be applied reliably?
   (d) What is a reasonable wavelength of the grain of sand and will it behave as a wave or as a particle?
   (e) What is the minimum uncertainty in its velocity if \( h = 6.67 \times 10^{-34} \) Js instead of \( 6.67 \times 10^{-10} \) Js.

9. An electron is held in orbit about a proton by electrostatic attraction.
   (a) Assume that an “orbiting electron wave” has the same energy an orbiting particle would have if at radius \( r \) and of momentum \( mv \). Write an expression for this energy.
   (b) If the electron behaves as a classical particle, it must obey \( F = ma \). Assuming circular orbit, apply \( F = ma \) to eliminate \( v \) in favor of \( r \) in the energy expression.
   (c) Suppose instead that the electron is an orbiting wave, and that the product of the uncertainties in radius \( r \) and momentum \( p \) is governed by an uncertainty relation of the form \( \Delta p \Delta r \approx \hbar \). Also assume that a typical radius of this orbiting wave is roughly equal to the uncertainty \( \Delta r \), and that a typical magnitude of the momentum is roughly equal to the uncertainty \( \Delta p \), so that the uncertainty relation becomes \( pr \approx \hbar \). Use this to eliminate \( v \) in favor of \( r \) in the energy expression.
   (d) Sketch on the same graph the expressions from parts (b) and (c).
   (e) Find the minimum possible energy for the orbiting electron wave, and the value of \( r \) to which it corresponds.

10. A beam of electrons is accelerated through a potential difference \( V \). The beam is then incident on a screen with two slits and a viewing screen is placed far away. An interference pattern is viewed on the screen as shown.
(a) Why is an interference pattern observed?
(b) Suppose the experiment is repeated with an accelerating potential of 0.5 V instead of V. How will the interference pattern change? Sketch the pattern clearly as well as the profile and compare with the original observations shown in Fig.(a) and (b).
(c) If V is reduced to 0.5 V, what is the effect on:
(i) K.E. of each electron reaching the slits?
(ii) Momentum of each electron reaching the slits.
(d) If the acceleration potential is still V (with the direction of V reversed of course), how would the interference profile change if one were to use protons instead of electrons?

11. An electron is confined in an infinite well of 30 cm width.
   (a) What is the ground-state energy?
   (b) In this state, what is the probability that the electron would be found within 10 cm of the left-hand wall?
   (c) If the electron instead has an energy of 1.0 eV, what is the probability that it would be found within 10 cm of the left-hand wall?
   (d) For the 1-eV electron, what is the distance between nodes and the minimum possible fractional decrease in energy?

12. A 50 eV electron is trapped in a finite well. How “far” (in eV) is it from being free if the penetration length of its wave function into the classically forbidden region is 1 nm?
13. Consider a particle that is bound inside an infinite well whose “floor” is sloping as shown.

\[ U(x) \rightarrow \infty \]

Sketch a plausible wave function when the energy is \( E_1 \) and when the energy is \( E_2 \).

14. In an infinite well, consider the 1st excited state, i.e., \( n = 2 \).

(a) What is the most probable position of the particle after a measurement has been made?

(b) What is the average position, \( \langle x \rangle \)?

15. The nuclear potential that binds protons and neutrons in the nucleus of an atom is often approximated by a square well. Imagine a proton confined in an infinite square well of length \( 10^{-5} \) nm, a typical nuclear diameter. Calculate the wavelength and energy associated with the photon that is emitted when the proton undergoes a transition from the first excited state (\( n = 2 \)) to the ground state (\( n = 1 \)). In what region of the electromagnetic spectrum does this wavelength belong?

16. Consider the potential energy barrier of length \( L \) and height \( V_0 \) as shown below. An electron is injected from the left. It has energy \( E < V_0 \).

\[ V(x) \]

Electron

Fig. (a)
(a) Write down the wavefunctions in regions I, II and III. These wavefunctions should include physically plausible terms. The Schrodinger equation (space part) is,

\[- \frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x) \psi(x) = E \psi(x).\]

(b) Write down the boundary conditions at \(x = 0\) and \(x = L\).

c) If a single electron is injected, will it be reflected from the wall at \(x = 0\)? Can it penetrate through the obstacle and be found at \(x > L\)?

d) Can the electron be “really”—I mean physically be found inside the region II? Use the uncertainty principle to answer this question.

e) Find the probability \(T\) that the incident electron from the far left is transmitted into region III.

(f) Now consider the Fig. (b) with, \(E < V_0\), \(E > W_0\), and \(V_0 > W_0\).

![Fig. (b)](image)

Using your result for part (e), find the transmission probability into region III.

(g) If the barrier in figure (a) is to act like a 50:50 beamsplitter, what are the required conditions on \(E, V_0\) and \(L\)?

17. The figure shows the potential energy landscape and the dashed line shows the energy of an electron trapped inside the infinite well.
The possible wave function is:

\[ x=0 \quad x=L \]

(a) (b) (c) (d) (e)

18.

A particle (photon) is injected from the left into the region I, it encounters a potential step of height \( V_0 \) and enters region II. The energy of the photon is \( 2V_0 \). What can you say about the ratio of the propagation speeds of the photon in regions I and II, \( v_I \) and \( v_{II} \)?

(a) \( v_I = v_{II} \).

(b) \( v_I < v_{II} \).

(c) \( v_I = \frac{v_{II}}{\sqrt{2}} \).

(d) \( v_I = \sqrt{2}v_{II} \).

(e) insufficient information is available to answer this question.
19. A free electron of energy $E$ has a de Broglie wavelength $\lambda = h/p = h/\sqrt{2mE}$ and speed $v$. In the presence of an electric field, it acquires a potential energy $-eu(x)$, where $u(x)$ is the potential. Hence the total energy changes, and the speed of the electron changes to $v'$. What is the value of refractive index $n = \frac{v}{v'}$?

(a) 1 (one).
(b) $\sqrt{E/u(x)}$.
(c) $\sqrt{E/(E - eu(x))}$.
(d) $\sqrt{(E - eu(x))/E}$.
(e) $\sqrt{eu(x)/E}$.

20. A particle is described by the wavefunction $\Psi(x, t) = e^{i(kx - \omega t)}$ and can be thought of a plane wave traveling along the $x$ axis. The real part at $t = 0$ is shown in the accompanying diagram. (The wavefunction extends from $-\infty$ to $\infty$ which of course we cannot show on paper.) Which of the following statements most accurately describes the probability of finding the particle.

(a) It is equally likely to find the particle anywhere along the $x$ axis.
(b) It is most likely to be found in the peaks of the wave.
(c) It is most likely to be found in the peaks or the troughs the wave.
(d) The position of the particle depends on when I make a measurement.
(e) I have no idea how to answer this question.

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21. An electron is trapped in a quantum dot of diameter $L$. The electron is in a potential well of depth $V_0$.

![Diagram of a quantum dot](image)

The energy values are approximately the same as the infinite well, $E = n^2 \frac{\pi^2 h^2}{2mL^2}$. A laser photon of energy $E_{\text{photon}}$ shines on the quantum dot in the ground state. What should be the minimum diameter if the electron is to always remain confined in the quantum dot?

22. The position wavefunction, $\psi(x)$ of a particle at some instant is given by,

$$\psi(x) = \frac{1}{L^2 + (x - x_o)^2/\alpha^2},$$

where $L, x_o, \alpha$ are constants. Which of the following expressions given below is a good approximate to the spread in the momentum, $\Delta p$? We are measuring spreads by FWHM (full width at half maximum), and $\Delta x \Delta p \geq \hbar$. (HINT: The wavefunction is maximum at $x = x_o$) and its profile is shown in the figure.

![Wavefunction profile](image)

(a) $\Delta p \sim \hbar/(\alpha L)$
(b) $\Delta p \sim \hbar/\alpha$
(c) $\Delta p \sim \hbar/(\alpha^2 x_o)$
23. The energy landscape for electrons in a metal is shown below in Fig.(a), while Fig.(b) shows the physical configuration.

![Energy Landscape Diagram]

The electrons have quantized energies and the most energetic electron is at the energy $E_F$ (called the Fermi energy). Now $E_F$ is below the vacuum level by an energy $\phi$ called the work function.

(a) In field emission, a large positive voltage $V_0$ is applied to a nearby metal tip as shown in Fig.(b). Sketch how the energy diagram shown in (a) is modified. How are electrons ejected from the metal?

(Don’t confuse voltage $V_0$ with potential energy $V$).

(b) Using your sketch of the modified potential energy diagram, find the minimum voltage $V_0$ required for the electron to tunnel out into the metal tip kept a distance $d$ away. The electron does not change its energy in the process.

(c) Assume that the critical distance for the tunneling of this kind is $d = 2.5$ nm and the work function is 4 eV. What is the electric field required for emission?

24. Advance an argument that there is no bound (=quantized) state in a half infinite well (shown in the diagram) unless the potential barrier $V_0$ is at least $\hbar^2/(8ma^2)$.
25. The equations for the reflection $R$ and transmission $T$ for light encountering a transparent film are essentially the same as a particle of energy $E$ seeing a potential discontinuity while $E$ always remains greater than the potential energy. Derive the value of the ratio between the wave numbers $k_1$ and $k_2$ if 50% of the light is always reflected at an interface.

26. A metal is held at zero voltage. The energy diagram at the metal-air interface is shown.

In thermionic emission, electrons are ejected from the metal surface because:

(a) The work function $\phi$ increases.

(b) The work function $\phi$ decreases.

(c) The potential energy seen by the electrons in the air slopes downward.

(d) Increasing temperature makes more electrons jump into unfilled levels increasing the fraction of electrons with thermal energy beyond $\phi$.

(e) The Fermi level $E_F$ decreases.
The figure shows the energy diagram for a metal in which electrons fill energy levels up to $E_F$. A thin insulating oxide layer separates the metal from a quantum dot with only ten quantized energy levels. The quantum dot is given a positive potential $V_0$ with respect to the metal, enabling an electron to tunnel across the oxide layer. Which one of these plots shows the correct behavior of the tunneling current $i$ from metal to the quantum dot. At $V_0 = 0$, $E_F$ is at the same energy as the $n = 7$ quantum level.
28. An electron is injected into a potential energy landscape from the left region I as shown below. It encounters a potential step. The energy of the electron is $E$ and $E < |V_0|$. If the electron is to emerge in region III with a faster speed, the appropriate potential step is given by which of the following?

(a) $E < V_0$
(b) $E > V_0$
(d) $E > V_0$
(c) $E V_0$

(e) The speed of the electron cannot increase.

29. Snell’s law of refraction determines the bending of light across an interface. For sure, electrons are also waves and can be refracted. The corresponding law for electrons is called Bethe’s law and is given by,

$$\frac{\sin \alpha}{\sin \beta} = \frac{v_2}{v_1},$$

where $\alpha$ is the angle of incidence measured from the normal to the interface, $\beta$ is the angle of refraction also measured from the normal, $v_1$ is the speed of electron in the incident medium and $v_2$ is the speed in the refracted medium. Now a beam of electrons is made to pass through two hollow cylinders with an applied voltage difference. Which of the following diagrams show the correct trajectory of electrons?
30. An electron starts off in the region $B$, trapped in a well. The potential energy $V(x)$ along position $x$ is shown.
Now suppose some time lapses. From a quantum viewpoint, which of the regions A or C, is the electron more probable to be found?

(a) Region A.

(b) Region C.

(c) Equal probability of being found in A and C.

(d) The electron absolutely cannot leave region B.

(e) None of the above.

31. A particle moving in a region of zero force encounters a precipice—a sudden drop in the potential energy to an extremely large negative value. What is the probability that it will “go over the edge”, i.e., it will enter the negative potential energy region?

(a) Almost zero.  
(b) Almost one.  
(c) \( \approx \frac{1}{2} \).  
(d) > \( \frac{1}{2} \).

(e) None of the above.

32. An electron is trapped inside a three-dimensional quantum dot. The energy is quantized in three dimensions according to,

\[
E_{n_x,n_y,n_z} = \frac{\pi^2 \hbar^2}{2m} \left( \frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2} \right),
\]

where \( a, b \) and \( c \) are the confining dimensions of the box (= dot) and \( n_x, n_y, n_z \) are the three quantum numbers, each one of them being a positive integer.

If \( a = b = c \), the energy difference between the ground and the first excited state is,
(a) $\frac{\pi^2\hbar^2}{2ma^2}$.
(b) $9\frac{\pi^2\hbar^2}{2ma^2}$.
(c) $3\frac{\pi^2\hbar^2}{2ma^2}$.
(d) $\frac{\pi^2\hbar^2}{ma^2}$.
(e) There are more than one “first excited states” all with different energies. Hence this question cannot be answered.

33. A free particle has a wave function,

$$\Psi(x, t) = Ae^{i(2.5 \times 10^{11}x - 2.1 \times 10^{13}t)},$$

where $x$ is in metres and $t$ is in seconds. What is the mass of the particle?

(a) Mass can only be determined if $A$ is known.
(b) 0.012 kg. (c) 0.11 kg.
(d) $5.7 \times 10^{-16}$ kg. (e) $1.7 \times 10^{15}$ kg.

34. An electron of energy 1 eV is trapped inside an infinite well of length 30 cm. What is the distance between two consecutive nodes of the electron’s wavefunction? (A node is a point where the wavefunction goes to zero.)

(a) There are no nodes in the electron’s wavefunction.
(b) The distance between consecutive nodes is zero.
(c) $1.25 \times 10^{-18}$ m.
(d) $6 \times 10^{-10}$ m.
(e) None of the above.

35. An electron is trapped in an infinite well of length $L$ and ground state energy $E_1$. At $t = 0$, the wavefunction is,

$$\Psi(x, 0) = \frac{1}{\sqrt{5L}} (\psi_1(x) + 2\psi_2(x)),$$

where $\psi_1(x)$ and $\psi_2(x)$ are normalized wavefunctions in the ground and first excited states. The wavefunction at $t = \frac{\pi\hbar}{E_1}$ is given by:
36. The potential energy profile in a certain region is shown.

A particle of energy $E$ exists inside this region. A sketch of the possible (real part) of the wavefunction is;
37. Suppose a particle is in the ground state with wavefunction $\psi_1(x)$. Which one of the following is the probability that the particle will be found in a narrow range between $x$ and $x + dx$.

(a) $|\psi_1(x)|^2 dx$. 
(b) $x|\psi_1(x)|^2 dx$. 
(c) $\int_x^{x+dx} x|\psi_1(x)|^2 dx$. 
(d) $\int_{-\infty}^{+\infty} x|\psi_1(x)|^2 dx$. 
(e) None of the above.

38. A free particle has a wavefunction $A(e^{ikx} + e^{-ikx})$ and energy $E$. $A$ is a normalization constant. Mark True or False against these statements.

(i) The probability density does not change with time.

(ii) The probability density is constant in space $x$.

(iii) The de Broglie wave associated with the particle is in fact a standing wave.
39. The uncertainty relationship for a particle moving in a straight line is $\Delta p \Delta x \geq \hbar/2$.

If the particle is moving in a circle with angular momentum $L$, the uncertainty relationship becomes:

(HINT: Distance becomes the arc length!)

(a) $\Delta L \Delta \theta \geq \hbar/2$.
(b) $\Delta L \Delta S \geq \hbar/2$.
(c) $\Delta L \Delta R \geq \hbar/2$.
(d) $\Delta L \Delta \theta \leq \hbar/2$.
(e) None of the above.

40. In a scanning tunneling microscope (STM), the tunneling probability of electrons from metal surface to a probe tip is proportional to $\exp(-2\alpha L)$, where $L$ is the tip-sample distance and $\alpha = 1 \text{ nm}^{-1}$ is the inverse of the penetration length.

If the tip moves closer to the surface by $\Delta L = 0.1 \text{ nm}$, the tunneling current,

(a) remains unchanged.
(b) increase by 22%.
(c) decrease by 22%.
(d) increase by 10%.
(e) decrease by 10%.

41. The Heisenberg uncertainty principle applies to photons as well as to material particles. Thus a photon confined to a small box of size $\Delta x$ necessarily has a large uncertainty in momentum and uncertainty in energy. Recall that for a photon $E = pc$.

(a) Estimate the uncertainty in energy for a photon confined to the tiny box of size $\Delta x$.

(b) If $\Delta E \sim E$, what is the effective mass of the photon?

(c) This mass can be extremely large, if $\Delta x$ is tiny. If $\Delta x$ is sufficiently small, the large mass can create a large gravitational field, sufficiently large to form a black hole. When this happens $\Delta x$ is called the Planck length, and this is when gravity and quantum mechanics become inter mixed. For a black hole, not even light can escape.

Consider a star of mass $M$ and radius $R$ as shown above. If an object is to be launched from the star’s surface so that it escapes the star’s gravitational pull, it needs a minimum velocity $v_{\text{esc}}$ called the escape velocity. Show that $v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$.

(d) If $v_{\text{esc}} = c$, nothing can escape from this star, not even light. If we were to replace the star of mass $M$ with a photon of the mass calculated in part (b), and confined to length $\Delta x$, and set $R = \Delta x$, calculate the Planck length in terms of $G$, $\hbar$ and $c$.

(e) If $G = 6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2$, find the numerical value of Planck’s length.

(f) What is the diameter of a proton (about 2 fm = $2 \times 10^{-15}$ m) in units of Planck’s length?
42. The radioactive decay of certain heavy nuclei by emission of an alpha particle is a result of quantum tunneling. Imagine an alpha particle moving around inside a nucleus, such as thorium (mass number= 232). When the alpha particles bounces against the surface of the nucleus, it meets a barrier caused by the attractive nuclear force. The dimensions of barrier vary a lot from one nucleus to another, but as representative numbers you can assume that the barrier’s width is \( L \approx 35 \text{ fm} = 10^{-15} \text{ m} \) and the average barrier height is such that \( V_0 - E \approx 5 \text{ MeV} \). Find the probability that an alpha hitting the nucleus surface will escape. Given that the alpha hits the nuclear surface about \( 5 \times 10^{21} \) times per second, what is the probability that it will escape in a day? The tunneling probability is \( T = e^{-2\alpha L} \) where \( \alpha = \sqrt{\frac{2m(V_0 - E)}{\hbar}} \) and \( L \) is the barrier length. (1 MeV= \( 10^6 \text{ eV} \)).
Solution

Answer 1:
The group velocity is given by,
\[ v_g = \frac{d\omega}{dk} = \left( \frac{d\omega}{dE} \right) \left( \frac{dE}{dk} \right). \]  
(1)

According to Einstein’s energy equation,
\[ E = h\omega. \]  
(2)

Classically, the energy of a slow moving particle is given by,
\[ E = \frac{1}{2}mv^2 = \frac{p^2}{2m}, \]  
(3)

where \( p = \) momentum of the particle = \( h/\lambda = (h/2\pi)(2\pi/\lambda) = \hbar k. \) Slow means \( v \ll c, \) implying that we can use the non-relativistic expressions. Therefore equation (38) becomes,
\[ E = \frac{\hbar^2 k^2}{2m}. \]  
(4)

From equation (37),
\[ \frac{dE}{d\omega} = \hbar \Rightarrow \frac{d\omega}{dE} = \frac{1}{\hbar}, \]  
(5)

while from equation (39) we obtain,
\[ \frac{dE}{dk} = \frac{2\hbar^2 k}{2m} = \frac{\hbar^2 k}{m}. \]  
(6)

Putting equations (40) and (41) into equation (42),
\[ v_g = \frac{1}{\hbar m} \frac{\hbar k}{m} = \frac{p}{m} = v_{\text{particle}} \text{, as required.} \]

Whereas the phase velocity is given by,
\[ v_p = \frac{\omega}{k} = \frac{E}{\hbar p} = \frac{E}{p} = \frac{p^2/2m}{p} = \frac{p}{2m} = \frac{v_{\text{particle}}}{2} \neq v_{\text{particle}}. \]

This clearly shows that for the atom, \( v_{\text{particle}} \) is exactly reproduced by the group velocity of the associated matter wave, whereas the phase velocity is half the particle velocity.
Answer 2:

For convenience, let’s denote $v_{\text{particle}} = v$. The energy and momentum of a relativistic particle (in this case, electron) are given by,

$$E = mc^2 = \frac{m_o c^2}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad (7)$$

$$p = mv = \frac{m_o v}{\sqrt{1 - \frac{v^2}{c^2}}}.$$  

The energy in terms of the relativistic momentum $p$ and the rest mass $m_o$ can be obtained from the expression we have now encountered several times,

$$E^2 = p^2 c^2 + m^2_o c^2$$

$$E = c \sqrt{p^2 + m^2_o c^2}.$$  

(8)

The corpuscular features (energy and momentum) of an electron are connected to its wave characteristics (wave frequency and number) by the relations,

$$E = h \omega \quad \text{and} \quad p = h k.$$  

Therefore the group and phase velocities will become,

$$v_g = \frac{d\omega}{dk} = \frac{dE}{dp}, \quad \text{and}$$

$$v_{\text{ph}} = \frac{E}{p} = \frac{E}{p}.$$  

From Equation (23) and (24), we find that $p^2 + m^2_o c^2 = \frac{m^2_o c^2}{1 - \frac{v^2}{c^2}}$. So, the phase velocity is

$$v_g = \frac{dE}{dp} = \frac{d}{dp} \left( c \sqrt{p^2 + m^2_o c^2} \right)$$

$$= \frac{pc}{\sqrt{p^2 + m^2_o c^2}}$$

$$= \frac{m_o v c / \sqrt{1 - \frac{v^2}{c^2}}}{E c}$$

$$= \frac{m_o v c^2}{E \sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \frac{m_o v c^2}{\sqrt{1 - \frac{v^2}{c^2}} \sqrt{1 - \frac{v^2}{c^2}} \ m_o c^2 .}$$

Hence $v_g = v$.  

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This shows that the speed of relativistic particle is equal to its group velocity.

Similarly, the phase velocity of relativistic particle can be calculated as

\[ v_{ph} = \frac{E}{p} = c \sqrt{\frac{p^2 + m^2 c^2}{p^2}} = c \sqrt{1 + \frac{m^2 c^2}{p^2}} = c \sqrt{1 + \frac{m^2 c^2}{m^2 v^2} \times \left(1 - \frac{v^2}{c^2}\right)} = c \sqrt{1 + \frac{c^2}{v^2} \left(1 - \frac{v^2}{c^2}\right)} = c \sqrt{\frac{c^2}{v^2}}. \]

Hence \( v_{ph} = c \left(\frac{c}{v}\right) \).

As \( c > v \), this means \( v_{ph} > c \), predicting that the phase velocity for the relativistic particle is greater than the speed of light \( c \). This appears to be a violation of the postulates of special theory of relativity. Actually, the phase velocity does not represent the physical velocity of the particle, rather it is the group velocity which represents the speed of propagation of the particle. Hence, the result that \( v_g \), and not \( v_p \), represents particle speeds, holds both in the relativistic and non-relativistic scenarios.

**Answer 3:**

(a) It is given that \( \Delta p \sim p \) and \( \Delta r \sim r \). When we consider small radii, the electron is present very close to the nucleus. Pushing the electron any closer to the nucleus results in increased energies. The electron may even gain enough energy to fly away from the nucleus. This is when the atom will ionize and hence the useful rule, “it is impossible to squish atoms”. Close to the nucleus, we are “rubbing shoulders” with the uncertainty principle.

According to this principle, the momentum of an electron confined within a given radius \( r \) is approximately given by \( p \sim \hbar/r \). (One could also use \( p \sim \hbar/(2r) \) without affecting the overall implications of the result. Remember that the uncertainty principle is an inequality!)

Therefore, when confined to a radius \( r \), the kinetic energy will be of the order,

\[ K.E = \frac{p^2}{2m} = \frac{\hbar^2}{2mr^2}. \]
Attempting to bring the nucleus any closer to the nucleus may result in extremely large kinetic energies, shooting the electron away.

(b) In the closest approach of the electron to the nucleus, the total energy of the hydrogen atom is,

\[
\text{Total Energy} = E = K.E + P.E
\]

\[
= \frac{\hbar^2}{2mr^2} - \frac{Ze^2}{4\pi\varepsilon_0 r}.
\]

The energy is minimum when \(dE/dr = 0\),

\[
\frac{dE}{dr} = \frac{h^2}{2m} \frac{d}{dr} \left( \frac{1}{r^2} \right) - \frac{Ze^2}{4\pi\varepsilon_0} \frac{d}{dr} \left( \frac{1}{r} \right)
\]

\[
= \frac{h^2}{2m} \left( \frac{-2}{r^3} \right) - \frac{Ze^2}{4\pi\varepsilon_0} \left( \frac{-1}{r^2} \right)
\]

\[
= -\frac{h^2}{mr^3} + \frac{Ze^2}{4\pi\varepsilon_0 r^2}.
\]

Setting this equal to zero,

\[
-\frac{h^2}{mr_{\text{min}}^3} + \frac{Ze^2}{4\pi\varepsilon_0 r_{\text{min}}^2} = 0
\]

\[
\frac{Ze^2}{4\pi\varepsilon_0 r_{\text{min}}^2} = \frac{h^2}{mr_{\text{min}}^3}
\]

\[
r_{\text{min}} = \frac{4\pi\varepsilon_0 \hbar^2}{mZe^2}.
\]

\[
= 0.53 \text{ Å}.
\]

This is the radius, \(r_{\text{min}}\), when the energy is minimum. The nucleus attracts the electron, so the electron prefers to exist close to the nucleus, but at the same time, the uncertainty principle does not let it come too close!
(c) The value of the radius calculated above is in excellent agreement with the radius of the smallest orbit \((n = 1)\) calculated from Bohr’s model.

**Answer 4:**

(a) Using the information provided in Question 1: \(\Delta p \sim p\) and \(\Delta r \sim r\), and using the uncertainty principle, the momentum of an electron confined within a radius \(r\) is approximately \(p \sim \hbar/r\). The total energy is,

\[
\text{Total Energy} = E = K.E + P.E = \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0r}.
\]

Ionization occurs when the energy of the electron approached zero, the energy of the vacuum state. We calculate the radius \(r_{\text{ion}}\) when \(E = 0\).

\[
\frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0r} = 0
\]

\[
\frac{\hbar^2}{2mr^2} = \frac{4\pi\epsilon_0r}{e^2}
\]

\[
r_{\text{ion}} = \frac{2\pi\epsilon_0\hbar^2}{me^2} = 0.24 \text{ Å}.
\]

The radius \(r_{\text{ion}}\) is smaller than the \(r_{\text{min}}\) calculated from the previous question, as we expect. Excessive pressure inside a planet can push the electron to this radius. At this point, the atoms will ionize and the planet will not be stable.

(b) The pressure is given as,

\[
P = -\frac{dE}{dV} = -\frac{dE}{dr} \frac{dr}{dV} \quad \text{using the chain rule.}
\]

Furthermore, we have,

\[
V = \frac{4}{3} \pi r^3
\]

\[
dV = 4\pi r^2 dr
\]

\[
\frac{dr}{dV} = \frac{1}{4\pi r^2}.
\]

Differentiating the energy expression from (9),

\[
\frac{dE}{dr} = \frac{\hbar^2}{2m} \left( \frac{-2}{r^3} \right) - \frac{1}{4\pi\epsilon_0} \frac{e^2}{r^2} \left( \frac{-1}{r^2} \right)
\]

\[
= -\frac{\hbar^2}{mr^3} + \frac{e^2}{4\pi\epsilon_0r^2}.
\]
We now substitute the value of the radius, \( r = r_{\text{ion}} \),

\[
\frac{dE}{dr} \bigg|_{r=r_{\text{ion}}} = -\frac{h^2}{2m} \left( \frac{1}{r_{\text{ion}}} \right)^3 + \frac{e^2}{4\pi\epsilon_0} \left( \frac{1}{r_{\text{ion}}} \right)^2 = -3.9 \times 10^7 \text{ J m}^{-1},
\]
resulting in the ionizing pressure,

\[
P_{\text{ion}} = -\frac{dE}{dr} \frac{1}{4\pi r_{\text{ion}}^2} = 5.2 \times 10^{13} \text{ Pa}.
\]

(c)

We assume a spherical planet of radius \( R \) and mass \( M \). We determine the parameters that result in ionizing pressures at the centre of the planet. First of all, we assume a constant density \( \rho \) of the planet throughout the interior. An estimate of the density is the proton mass divided by the volume of the atom,

\[
\rho = \frac{m_p}{\frac{4}{3}\pi r_{\text{ion}}^3} = 2.8 \times 10^4 \text{ kg m}^{-3}.
\]

The pressure exerted by a fluid of length \( R \) at its base is given by \( \rho gR \). However, the value of \( g \) on this planet is unknown, but from Newton’s law of gravitation, we know that \( g = GM/R^2 \). Therefore,

\[
P_{\text{ion}} = \rho g R = \frac{\rho G M}{R} \quad (11)
\]
\[
\Rightarrow R = \frac{\rho G M}{P_{\text{ion}}} = 3.5 \times 10^{-20} M \text{ m}. \quad (12)
\]

Now the density \( \rho \) can also be equated to the mass of the planet divided by its volume,

\[
\rho = 2.8 \times 10^4 \text{ kg m}^{-3} = \frac{M}{\frac{4}{3}\pi R^3} \quad (13)
\]
\[
\Rightarrow M = \frac{4}{3} \pi \rho R^3. \quad (14)
\]
Inserting the value of $M$ into (12) and then back substituting results in,

\[ M = 4 \times 10^{26} \text{ kg}, \]
\[ R = 1.6 \times 10^{7} \text{ m}. \]

The measured mass and radius of Jupiter are $1.9 \times 10^{27} \text{ kg}$ and $7 \times 10^{7} \text{ m}$ (values taken from Wikipedia).

**Answer 8:**

We are given that,

- Mass of grain = $m = 1 \text{ mg} = 10^{-6} \text{ kg}$
- Uncertainty in position = $\Delta x = 550 \text{ nm} = 550 \times 10^{-9} \text{ m}$.

(a) Uncertainty in velocity can be calculated by calculating uncertainty in its momentum. According to uncertainty principle, the minimum uncertainty is approximately,

\[ \Delta x \Delta p \geq \frac{\hbar}{2} \Rightarrow \Delta p \geq \frac{\hbar}{2\Delta x} = \frac{1.05 \times 10^{-34}}{2 \times 550 \times 10^{-9} \text{ m}} = 9.65 \times 10^{-29} \text{ kg m s}^{-1}. \]

$\Delta p$ is small because $\hbar$ is small. Now the uncertainty in speed is calculated as,

\[ \Delta p = m \Delta v \]
\[ \Delta v = \frac{\Delta p}{m} = \frac{9.7 \times 10^{-29} \text{ kg m s}^{-1}}{10^{-6} \text{ kg}} = 9.65 \times 10^{-23} \text{ m s}^{-1}. \]

For macroscopic particle $\Delta v \geq \hbar/2(\Delta x)m$ is small because of the very small $\hbar/m$ ratio. $\Delta v$ becomes significant only if $\hbar$ were large or the mass $m$ decreases. Small $\hbar$ and large $m$ makes the macroscopic classic world “undisturbed” by quantum uncertainties!

(b)

\[ \Delta t \approx \frac{1 \mu \text{m}}{\Delta v} = \frac{10^{-6}}{9.65 \times 10^{-23}} \text{s} = 0.1 \times 10^{17} \text{ s} \approx 3 \text{ billion years!} \]
The uncertainty in velocity is really really small! An observer would require 3 billion years to notice the grain of sand, supposedly at rest, at a position 1 µm away from its original position. The current age of the solar system is approximately 5 billion years.

(c) Yes uncertainties are extremely small. No device has ever been built, and may never be built that can measure these small velocities. We can safely apply classical mechanics to a grain of sand; there is effectively no uncertainty in position or in momentum. Furthermore, a precision as fine as $10^{-22}$ m/s is never required in classical mechanics.

(d) \[
\lambda = \frac{h}{mv} = \frac{h}{p}.
\]

Now what momentum should I choose? The uncertainty principle dictates a $\Delta p \sim 9.7 \times 10^{-29}$ kg m s$^{-1}$. The momentum could therefore have any value between, approximately $-\Delta p/2$ and $\Delta p/2$. Let’s choose an extreme value, $p \sim \Delta p/2 \sim 5 \times 10^{-29}$ kg m s$^{-1}$. Therefore,

\[
\lambda \sim \frac{6.67 \times 10^{-34}}{5 \times 10^{-29}} \sim 1.3 \times 10^{-5} \text{ m}.
\]

This is such a small wavelength compared to apparatus we might use for macroscopic objects, that for all practical purposes, the grain of sand acts like a particle!

(e) $\Delta v$ would be $9.65 \times 10 \approx 96$ m/s, if $h$ were this large. This is a huge uncertainty. We are “saved” by the exceedingly small value of $h$

**Answer 9:**

(a) Total energy of an orbiting particle in terms of its kinetic energy and electrostatic potential energy is given by,

\[
E_{\text{total}} = \text{K.E.} + \text{P.E.}.
\]

\[
E_{\text{total}} = \frac{1}{2}mv^2 + \left[ -\frac{ke^2}{r} \right]
\]

\[
E_{\text{total}} = \frac{1}{2}mv^2 - \frac{ke^2}{r},
\]

where $k = 1/4\pi\varepsilon_0$. Therefore total energy of the particle will become,

\[
E_{\text{total}} = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\varepsilon_0 r}.
\]

Hence the energy of “orbiting electron wave” is also,

\[
E = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\varepsilon_0 r}.
\]
(b) Since electron is orbiting in a circular orbit, its centripetal acceleration in its orbit is \(\left(\frac{v^2}{r}\right)\), while electrostatic force on the electron is \(\left(\frac{ke^2}{r^2}\right)\), thus,

\[
\begin{align*}
F &= ma \\
\frac{ke^2}{r^2} &= m\left(\frac{v^2}{r}\right) \\
\Rightarrow v^2 &= \frac{ke^2}{mr} \\
v^2 &= \frac{e^2}{4\pi\varepsilon_0 mr}.
\end{align*}
\]

Use this value of \(v^2\) in equation (42),

\[
E_{\text{classical particle}} = \frac{1}{2}m\left(\frac{e^2}{4\pi\varepsilon_0 mr}\right) - \frac{e^2}{4\pi\varepsilon_0 r}
= \frac{1}{2}\left(\frac{e^2}{4\pi\varepsilon_0 r}\right) - \frac{e^2}{4\pi\varepsilon_0 r}
= \frac{e^2}{8\pi\varepsilon_0 r} - \frac{e^2}{4\pi\varepsilon_0 r}
= -\frac{e^2}{8\pi\varepsilon_0 r}.
\]

The negative electrostatic potential energy is always of greater magnitude than the positive kinetic energy, so the total energy strictly decreases as \(r\) decreases. Hence there is no minimum energy. In the accompanying figure, course \(A\) corresponds to the energy of the classical particle, whose energy decreases as \(r\).

(c) Now assuming \(pr = \hbar\), we have \(p = \hbar/r\) or \(v = \hbar/mr\). Therefor equation (42) becomes,

\[
E_{\text{matter wave}} = \frac{1}{2}m\left(\frac{\hbar}{mr}\right)^2 - \frac{e^2}{4\pi\varepsilon_0 r}
= \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\varepsilon_0 r}.
\]

In this case as \(r\) decreases, and the wave become more compact, the likely speed increases. The kinetic energy increases faster than the potential decreases, and the total energy at some point must increase. Hence applying uncertainty principle there is a turning point \(A\) in the curve labelled \(B\).

(d) The two plots are shown in the figure.
While the energy of a classical particle would monotonically decrease as $r$ decreases, the energy of the matter wave reaches a minimum, and then increases.

(e) The minimum possible energy for the orbiting electron wave can be calculated by setting the derivative of energy with respect to $r$, to zero.

$$E_{\text{matter wave}} = \frac{\hbar^2}{2mr^2} - \frac{e^2}{4\pi\epsilon_0 r}$$

$$\frac{dE_{\text{matter wave}}}{dr} = -\frac{\hbar^2}{mr^3} + \frac{e^2}{4\pi\epsilon_0 r^2} = 0$$

$$\Rightarrow r = \frac{4\pi\epsilon_0 \hbar^2}{me^2}$$

$$= \frac{9 \times 10^9 \text{Nm}^2/\text{C}^2 \times (1.055 \times 10^{-34} \text{Js})^2}{9.11 \times 10^{-31} \times (1.6 \times 10^{-19} \text{C})^2}$$

$$= 5.3 \times 10^{-11} \text{ m.}$$

This turns out to be astoundingly close to the Bohr radius calculated earlier in class. Inserting this value of $r$ and other constants will give energy for matter wave as follows.

$$E_{\text{matter wave}} = -13.6 \text{ eV.}$$

The energy happens to equal the correct, experimentally determined value, and the radius is indeed the most probable radius at which the electron would be found. That these agree so closely is an accident; many approximations have been made. Nevertheless, the uncertainty principle does impose a lower limit on the energy, and it is no accident that the value we obtained is of the correct order of magnitude.

**Answer 10:**

(a) The electrons have a quantum field $\psi$ as they are present in the region between the
screens. This field is a superposition of two fields, \( \psi_1 \) and \( \psi_2 \) corresponding to two electrons ejecting from either of the slits. The phase difference between \( \psi_1 \) and \( \psi_2 \) results in these field interfering causing an interference pattern.

**N.B.** I don’t like the expression “wavefunction splitting” or “electron splitting”. “Electrons interfering with themselves” is fine.

(b) \( V \) is halved, so energy \( eV = \frac{p^2}{2m} \) is halved. Hence \( p^2 \) is halved or \( p \) is reduced \( \sqrt{2} \) times. Since \( p = \frac{h}{\lambda} \), \( \lambda \) increases \( \sqrt{2} \) times and \( k = \frac{2\pi}{\lambda} \) decreases \( \sqrt{2} \) times. If the voltage were \( V \) gave a wavenumber \( k \), then a voltage \( V/2 \) gives a wavenumber \( k/\sqrt{2} \). The interference pattern is proportional to \( \cos^2(kd \sin \theta/2) \). Hence after the halving, the pattern is proportional to \( \cos^2((kd \sin \theta)/2\sqrt{2}) \).

Let \( (kd \sin \theta)/2 = \alpha \). With voltage \( V \), intensity pattern \( \propto \cos^2(\alpha) \). With voltage \( V/2 \), intensity pattern \( \propto \cos^2(\alpha/\sqrt{2}) = \cos^2(0.707\alpha) \).

Now a minimum is observed when \( \cos^2(\alpha) = 0 \quad \Rightarrow \alpha = \pi/2 \), when voltage is \( V \). Whereas, if the voltage is \( V/2 \), the minimum appears at \( \cos^2(0.707\alpha) = 0 \quad \Rightarrow 0.707\alpha = \pi/2 \quad \Rightarrow \alpha = \pi/2 \times 0.707 \), which is larger than the previous case. Hence the fringe width increases \( \sqrt{2} = 1.411 \) times.

![Diagram](image)

(c)

(i) \( E = \frac{p^2}{2m} = eV \), if \( V \) is halved \( E \) is halved.

(ii) If \( V \) is halved \( p^2 \) is halved, so \( p \) is reduced \( \sqrt{2} \) times.

(d) The protons gain the same energy as electrons, because they carry the same charge. But protons are heavier than electrons, so their momentum is larger \( (p^2 = 2mE) \), hence \( \lambda \) is

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shorter and \( k = 2\pi/\lambda \) is larger. Since energy pattern \( \propto \cos^2((kd\sin \theta)/2) \), a large \( k \) would result in rapid spatial variation in bright and dark fringes which will therefore be squeezed close together.

**Answer 11:**

(a) For an infinite square well, the energy is,

\[
E_n = \frac{\pi^2 \hbar^2 n^2}{2mL^2}.
\]

For the ground state, \( n = 1 \) and the corresponding energy is,

\[
E_1 = \frac{\pi^2 \hbar^2}{2mL^2} = \frac{\pi^2 (1.054 \times 10^{-34} \text{ J sec})^2}{2(9.1 \times 10^{-31} \text{ kg})(0.3 \text{ m})^2} = 6.71 \times 10^{-37} \text{ J}.
\]

(b) The wavefunction for an infinite square well is,

\[
\psi_n = \sqrt{\frac{2}{L}} \sin \left( \frac{n\pi x}{L} \right)
\]
\[
\psi_1 = \sqrt{\frac{2}{L}} \sin \left( \frac{\pi x}{L} \right).
\]
The probability of finding the electron within 10 cm of the left-hand wall is,

\[ P(0 < x < 0.1 \text{ m}) = \int_0^{0.1} \psi^*(x)\psi(x)dx \]

\[ = \int_0^{0.1} |\psi(x)|^2dx \]

\[ = \frac{2}{L} \int_0^{0.1} \sin^2(\frac{\pi x}{L})dx \]

\[ = \frac{2}{L} \int_0^{0.1} \left(1 - \cos(2\frac{\pi x}{L})\right)\frac{1}{2}dx \]

\[ = \frac{1}{L} \int_0^{0.1} \left(1 - \cos(2\frac{\pi x}{L})\right)dx \]

\[ = \frac{1}{L} \left[1 - \frac{\sin(2\frac{\pi x}{L})}{2\pi/L}\right]_0^{0.1} \]

\[ = 0.21. \]

(c) If the electron has 1.0 eV of energy, then,

\[ \frac{\pi^2\hbar^2n^2}{2mL^2} = 1.6 \times 10^{-19} \text{ J} \]

\[ n^2 = \frac{2mL^2(1.6 \times 10^{-19} \text{ J})}{\pi^2\hbar^2} = \frac{2(9.11 \times 10^{-31} \text{ kg})(0.3 \text{ m})^2(1.6 \times 10^{-19} \text{ J})}{\pi^2(1.054 \times 10^{-34} \text{ J sec})^2} \]

\[ = 2.38 \times 10^{17} \]

\[ n = 4.88 \times 10^8. \]

With this energy of electron, the probability of finding it within 10 cm of left-hand wall is,

\[ P(0 < x < 0.1 \text{ m}) = \int_0^{0.1} \psi^*(x)\psi(x)dx \]

\[ = \frac{2}{L} \int_0^{0.1} \sin^2(n\pi x/L)dx \]

\[ = \frac{2}{L} \int_0^{0.1} \left(1 - \cos(2n\pi x/L)\right)\frac{1}{2}dx \]

\[ = \frac{1}{L} \int_0^{0.1} \left(1 - \cos(2n\pi x/L)\right)dx \]

\[ = \frac{1}{L} \left[1 - \frac{\sin(2n\pi x/L)}{2n\pi/L}\right]_0^{0.1} \]

\[ = 0.33. \]
(d) We know that,

\[ L = n \frac{\lambda}{2} \]
\[ \lambda = \frac{2L}{n} \]

Now the distance between the nodes is,

\[ \frac{\lambda}{2} = \frac{L}{2} = \frac{0.3}{4.8 \times 10^8} = 6.15 \text{ Å}. \]

The maximum possible fractional decrease in energy is thus,

\[ \frac{\Delta E}{E} = \frac{E_n - E_{n-1}}{E_n} = \frac{n^2 - (n-1)^2}{n^2} = \frac{2}{n} \frac{1}{n^2}, \]

since \( n = 4.8 \times 10^8 \), the minimum fractional decrease in energy is,

\[ \frac{\Delta E}{E} = 4.1 \times 10^{-9} \text{ J}. \]

**Answer 12:**

\[
\begin{align*}
\text{E} & \quad \text{---} \quad \text{---} \\
\( \delta \) & \quad \text{---} \quad \text{---} \\
\text{U_0} & \quad \text{---} \quad \text{---} \\
\text{x=0} & \quad \text{---} \quad \text{---} \\
\text{x=L} & \quad \text{---} \quad \text{---}
\end{align*}
\]

The penetration depth \( \delta \) is given by,

\[ \delta = \frac{\hbar}{\sqrt{2m(U_0 - E)}} = 1 \times 10^{-9} \text{ m} \]

\[ 2m(U_0 - E) = \frac{\hbar^2}{(1 \times 10^{-9} \text{ m})^2} \]

\[ U_0 - E = \frac{\hbar^2}{2m(1 \times 10^{-9} \text{ m})^2} = 38.2 \text{ meV}. \]
Answer 13:
The plausible wavefunctions are shown in the Figure.

![Wavefunction Diagram](image)

The wavefunction $\psi_1(x)$ corresponds to energy $E_1$ and $\psi_2(x)$ corresponds to energy $E_2$. If $E > U(x)$, the wavefunction is $\propto \exp\left(\frac{i\sqrt{2m(E-U(x))}}{\hbar}\right)$, which is oscillatory. Larger the value of $E - U(x)$, higher the value of $k = \sqrt{\frac{2m(E-U(x))}{\hbar}}$ and shorter the wavelength. If $E < U(x)$, the wavefunction decays (damps). Note that the wavelength for $\psi_1(x)$ is not uniform, rather the wavelength increases, $k$ decreases as $E - U(x)$ decreases in going from left to right.

Answer 14:
The wavefunction for an infinite well is,

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

For first excited state, $n = 2$,

$$\psi_2(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right)$$

and the probability is,

$$P_2(x) = |\psi_2(x)^*\psi_2(x)| = \frac{2}{L} \sin^2\left(\frac{2\pi x}{L}\right).$$

To find the most probable position, we have to maximize $P_2(x)$.

$$\frac{dP_2(x)}{dx} = 2\left(\frac{2}{L}\right) \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{2\pi x}{L}\right) \left(\frac{2\pi}{L}\right)$$

$$= 2\pi \left(\frac{4}{L^2}\right) \sin\left(\frac{2\pi x}{L}\right) \cos\left(\frac{2\pi x}{L}\right).$$
The quantity \( \frac{dP_2(x)}{dx} = 0 \) when \( x = 0, L/4, L/2, 3L/4, L \), but when \( x = 0, L/2, L \), the wavefunction \( \psi_2(x) = 0 \).

Thus at \( x = 0, L/2, L \), the probability of finding the particle is zero. The most probable positions are \( x = L/4 \), and \( x = 3L/4 \).

(b) The average position is given by the following,

\[
\langle x \rangle = \int_0^L x \psi_2^*(x) \psi_2(x) dx
= \int_0^L x |\psi_2(x)|^2 dx
= \frac{2}{L} \int_0^L x \sin^2(2\pi x/L) dx
= \frac{2}{2L} \int_0^L x (1 - \cos(4\pi x/L)) dx
= \frac{1}{L} \int_0^L x dx - \frac{1}{L} \int_0^L x \cos(4\pi x/L) dx
= \frac{L}{2} - 0
= \frac{L}{2}.
\]

**Answer 15:**

In a square well, the energy that corresponds to \( n \)’th energy level is,

\[ E_n = \frac{\pi^2 h^2 n^2}{2mL^2}. \]

When a proton undergoes a transition from the first excited state \( (n = 2) \) to the ground state \( (n = 1) \), the energy of emitted photon is,

\[
\Delta E_{2 \to 1} = \frac{\pi^2 h^2}{2mL^2} (2^2 - 1^2)
= \frac{3\pi^2 h^2}{2mL^2}
= \frac{3\pi^2 (1.054 \times 10^{-34} \text{ Js})^2}{2(1.67 \times 10^{-27} \text{ kg})(10^{-14} \text{ m})^2}
= 9.8 \times 10^{-15} \text{ J}.
\]

The wavelength of emitted photon is,

\[
\lambda = \frac{hc}{\Delta E}
= \frac{(6.63 \times 10^{-34} \text{ Js})(3 \times 10^8 \text{ m/s})}{9.8 \times 10^{-15} \text{ J}}
= 2 \times 10^{-11} \text{ m}.
\]
Answer 16:

(a) The time independent Schrödinger equation is,

\[ -\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x). \]

For Region I:

For region I the Schrödinger equation, with \( V = 0 \), is,

\[
\begin{align*}
-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} &= E\psi(x) \\
\frac{d^2 \psi(x)}{dx^2} + \frac{2mE}{\hbar^2} \psi(x) &= 0 \\
\left( \frac{d^2}{dx^2} + \frac{2mE}{\hbar^2} \right) \psi(x) &= 0 \\
(D^2 + k^2)\psi(x) &= 0,
\end{align*}
\]

where \( k = \frac{2mE}{\hbar^2} \).

\[ \Rightarrow D^2 = -k^2 \]

\[ D = \pm ik, \]

which leads to the solution,

\[ \psi(x) = Ae^{ikx} + Be^{-ikx}. \]

the first term on the right is a forward propagating wave and the second term is a backward propagating wave.

For Region II: \( V = V_0 \) and \( E < V_0 \)

The Schrödinger equation for this region becomes,

\[
\begin{align*}
-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + (V_0 - E)\psi(x) &= 0 \\
\left( -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + (V_0 - E) \right) \psi(x) &= 0 \\
\left( \frac{d^2}{dx^2} - \frac{2m}{\hbar^2} (V_0 - E) \right) \psi(x) &= 0 \\
(D^2 - \alpha^2)\psi(x) &= 0
\end{align*}
\]

\[ D = \pm \alpha, \]

where \( \alpha^2 = \frac{2m(V_0 - E)}{\hbar^2} \), is a positive constant.
Hence wave function for region II is,

$$\psi_{II}(x) = Ce^{\alpha x} + De^{-\alpha x}$$  \hspace{1cm} (17)

Since the exponents $\alpha$ are real, $\psi_{II}(x)$ does not represent an oscillatory function, rather it is a decreasing function.

**For Region III:** $V=0$

Since the wavevector of region III is same as that of region I, hence the solution becomes,

$$\psi(x) = Ee^{ikx} + Fe^{-ikx}.$$  

Since the wave does not seen any obstacle in region III, it cannot be reflected implying $F = 0$. The wavefunction therefore, is,

$$\psi_{III}(x) = Ee^{ikx}.$$  

**(b)** The boundary condition at the edges of the barrier are that $\psi(x)$ and $\frac{d\psi(x)}{dx}$ must be continuous at both edges.

**At x=0:**

$$\psi_I(x = 0) = \psi_{II}(x = 0)$$

$$\Rightarrow A + B = C + D$$  \hspace{1cm} (18)

and

$$\left. \frac{d\psi_I}{dx} \right|_{x=0} = \left. \frac{d\psi_{II}}{dx} \right|_{x=0}$$

$$ikA - ikB = \alpha C - \alpha D$$

$$ik(A - B) = \alpha(C - D)$$

$$A - B = -i\frac{\alpha}{k}(C - D).$$  \hspace{1cm} (19)

**At x=L:**

$$\psi_{II}(x = L) = \psi_{III}(x = L)$$

$$\Rightarrow Ce^{\alpha L} + De^{-\alpha L} = Ee^{ikL}$$  \hspace{1cm} (20)

and

$$\left. \frac{d\psi_{II}}{dx} \right|_{x=L} = \left. \frac{d\psi_{III}}{dx} \right|_{x=L}$$

$$\alpha Ce^{kL} - \alpha De^{-\alpha L} = ike^{ikL}$$

$$Ce^{\alpha L} - De^{-\alpha L} = i\frac{k}{\alpha} E e^{ikL}.$$  \hspace{1cm} (21)

Equations (37), (38), (39) and (40) are the required boundary conditions.
(c) If a single electron is injected with energy less than the barrier height \((E < V_0)\), classically the particle cannot penetrate through the barrier, it will be reflected completely. However quantum mechanics tell us that the particle has certain probability to go through the barrier, and this is when the transmission coefficient is non-zero.

(d) In region II the total energy is less than the potential energy, which means that the particle appears to possess negative kinetic energy here. From equation (17) on page 3, \(1/\alpha\) represents a penetrating length \(\delta\),

\[
\delta = \frac{1}{\alpha} = \frac{\hbar}{\sqrt{2m(V_0 - E)}},
\]
which characterizes a realm of possibilities of position \(\Delta x\). Now using the uncertainty principle \(\Delta x\Delta p \sim \hbar\)

\[
\Delta p \sim \frac{\hbar\sqrt{2m(V_0 - E)}}{\hbar} = \sqrt{2m(V_0 - E)} \sim p.
\]

The momentum \(p\) corresponds to an energy \(p^2/2m = V_0 - E\) and if we assume that the uncertainty in energy \(\Delta E\) is of the same order as the energy, the \(\Delta E \sim V_0 - E\).

As you immediately recognize, \(V_0 - E\) is of the same order as the energy gap that localizes the particle to the region \(\delta\). If the uncertainty is as large as the gap, there is no guarantee that the particle can be localized.

(e) Transmission probability \(T\) is given by,

\[
T = \frac{\text{Prob. that the particle crosses right boundary per unit time}}{\text{Prob. that the particle crosses left boundary per unit time}}
\]

\[
= \frac{|\text{prob/time}|_{x=L}}{|\text{prob/time}|_{x=0}}
\]

\[
= \frac{|\text{prob/length} \times \text{length/time}|_{x=L}}{|\text{prob/length} \times \text{length/time}|_{x=0}}
\]

\[
= \frac{|\psi(x)|^2_{x=L} \times \nu}{|\psi(x)|^2_{x=0} \times \nu}
\]

\[
= \frac{|\psi(x)|^2_{x=L}}{|\psi(x)|^2_{x=0}}.
\]

In the given case, \(|\psi(x)|^2_{x=L} = |E|^2\) and \(|\psi(x)|^2_{x=0} = |A|^2\). therefore transmission probability is,

\[
T = \frac{|E|^2}{|A|^2}.
\]
In order to calculate $|E|^2$ and $|A|^2$ we will use the four boundary conditions derived in part (b). Add equation (37) and (38),

$$2A = C \left( 1 - \frac{i \alpha}{k} \right) + D \left( 1 + \frac{i \alpha}{k} \right),$$

$$A = \frac{C}{2} \left( 1 - \frac{i \alpha}{k} \right) + \frac{D}{2} \left( 1 + \frac{i \alpha}{k} \right). \quad (22)$$

Next adding equation (39) and (40) we obtain,

$$2C e^{\alpha L} = E e^{ikL} \left( 1 + \frac{k}{\alpha} \right)$$

$$C = \frac{E e^{ikL}}{2e^{\alpha L}} \left( 1 + \frac{k}{\alpha} \right). \quad (23)$$

Now subtracting equation (40) from (39) results in,

$$2D e^{-\alpha L} = E e^{ikL} \left( 1 - \frac{k}{\alpha} \right)$$

$$D = \frac{E e^{ikL}}{2e^{-\alpha L}} \left( 1 - \frac{k}{\alpha} \right). \quad (24)$$

Using values of $C$ and $D$ from equations (23) and (24) in equation (41) yields,

$$A = \frac{1}{2} \left( \frac{E e^{ikL}}{2e^{\alpha L}} \right) \left( 1 + \frac{k}{\alpha} \right) \left( 1 - \frac{i \alpha}{k} \right) + \frac{1}{2} \left( \frac{E e^{ikL}}{2e^{-\alpha L}} \right) \left( 1 - \frac{k}{\alpha} \right) \left( 1 + \frac{i \alpha}{k} \right)$$

$$= \frac{E e^{ikL}}{4} \left[ e^{-\alpha L} \left( 1 + \frac{i k}{\alpha} - i \frac{\alpha}{k} + 1 \right) + e^{\alpha L} \left( 1 + i \frac{\alpha}{k} - i \frac{k}{\alpha} + 1 \right) \right]$$

$$= \frac{E e^{ikL}}{4} \left[ e^{-\alpha L} \left( 2 + i \frac{(\alpha^2 - k^2)}{k \alpha} \right) + e^{\alpha L} \left( 2 - i \frac{(k^2 - \alpha^2)}{k \alpha} \right) \right],$$

and after some further rearrangement,

$$A = \frac{E e^{ikL}}{4} \left[ 2(e^{\alpha L} + e^{-\alpha L}) + i \frac{(\alpha^2 - k^2)}{k \alpha} (e^{\alpha L} - e^{-\alpha L}) \right]$$

$$= E e^{ikL} \left[ \frac{(e^{\alpha L} + e^{-\alpha L})}{2} + i \frac{(\alpha^2 - k^2)}{2k \alpha} \left( \frac{e^{\alpha L} - e^{-\alpha L}}{2} \right) \right].$$

Using $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2},$

$$A = E e^{ikL} \left( \cosh(\alpha L) + i \frac{(\alpha^2 - k^2)}{2k \alpha} \sinh(\alpha L) \right). \quad (25)$$

Now taking the complex conjugate of equation (25),

$$A^* = E^* e^{-ikL} \left( \cosh(\alpha L) - i \frac{(\alpha^2 - k^2)}{2k \alpha} \sinh(\alpha L) \right). \quad (26)$$
and finally multiplying equations (25) and (26) yields,

\[ A^* A = E^* e^{-ikL} \left( \cosh(\alpha L) - i \frac{(\alpha^2 - k^2)}{2k \alpha} \sinh(\alpha L) \right) \cdot E e^{ikL} \left( \cosh(\alpha L) + i \frac{(\alpha^2 - k^2)}{2k \alpha} \sinh(\alpha L) \right) \]

\[ |A|^2 = |E|^2 \left( \cosh^2(\alpha L) + \frac{(\alpha^2 - k^2)^2}{4k^2 \alpha^2} \sinh^2(\alpha L) \right) \]

\[ \frac{|A|^2}{|E|^2} = \left( 1 + \sinh^2(\alpha L) + \frac{(\alpha^2 - k^2)^2}{4k^2 \alpha^2} \sinh^2(\alpha L) \right) \]

\[ = 1 + \left( 1 + \frac{(\alpha^2 - k^2)^2}{4k^2 \alpha^2} \right) \sinh^2(\alpha L) \]

\[ = 1 + \frac{(\alpha^2 + k^2)^2}{4k^2 \alpha^2} \sinh^2(\alpha L). \]

Substitute values of \( k \) and \( \alpha \) we obtain the desired result.

\[ \frac{|A|^2}{|E|^2} = T^{-1} = 1 + \frac{2m(V_0 - E)}{4(\frac{2m(V_0 - E)}{\hbar^2} + \frac{2mE}{\hbar^2})} \sinh^2 \left( \frac{L}{\hbar} \sqrt{2m(V_0 - E)} \right) \]

\[ T^{-1} = 1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 \left( \frac{L}{\hbar} \sqrt{2m(V_0 - E)} \right) \]

\[ T = \left[ 1 + \frac{V_0^2}{4E(V_0 - E)} \sinh^2 \left( \frac{L}{\hbar} \sqrt{2m(V_0 - E)} \right) \right]^{-1}. \]

Shown is a graph of \( T \) versus \( E \) for \( V_0 = 10 \times 1.6 \times 10^{-19} \text{ J} = 10 \text{ eV} \) and \( E \) in the range of 0 to \( 30 \times 1.6 \times 10^{-19} \text{ J} (= 30 \text{ eV}) \). The length of the barrier \( L \) is chosen as 1Å. Clearly as \( E \) increases, the transmission \( T \) goes up.
Wave functions for the three regions are,
\[ \psi_I(x) = Ae^{ikx} + Be^{-ikx}, \quad \text{where} \ k = \frac{\sqrt{2mE}}{\hbar} \]
\[ \psi_{II}(x) = Ce^{\alpha x} + De^{-\alpha x}, \quad \text{where} \ \alpha = \frac{\sqrt{2m(V_0 - E)}}{\hbar} \]
\[ \psi_I(x) = Ee^{i\beta x}, \quad \text{where} \ \beta = \frac{\sqrt{2m(E - W_0)}}{\hbar} \]

The boundary conditions at \( x = 0 \) are,
\[ \psi_I(0) = \psi_{II}(0) \]
\[ A + B = C + D \quad (27) \]
and \( \psi'_I(0) = \psi'_{II}(0) \)
\[ A - B = -i\frac{\alpha}{k}(C - D), \quad (28) \]

while the boundary conditions at \( x = L \) are,
\[ \psi_{II}(L) = \psi_{III}(L) \]
\[ Ce^{\alpha L} + De^{-\alpha L} = Ee^{i\beta L} \quad (29) \]
and \( \psi'_I(L) = \psi'_{II}(L) \)
\[ Ce^{\alpha x} - De^{-\alpha L} = i\frac{\beta}{\alpha}Ee^{i\beta L}. \quad (30) \]

Adding equations (27) and (28),
\[ 2A = C\left(1 - i\frac{\alpha}{k}\right) + D\left(1 + i\frac{\alpha}{k}\right) \]
\[ A = \frac{C}{2}\left(1 - i\frac{\alpha}{k}\right) + \frac{D}{2}\left(1 + i\frac{\alpha}{k}\right), \quad (31) \]
and add equation (29) and (30),
\[ 2Ce^{\alpha L} = Ee^{i\beta L}\left(1 + i\frac{\beta}{\alpha}\right) \]
\[ C = \frac{Ee^{i\beta L}}{2e^{\alpha L}}\left(1 + i\frac{\beta}{\alpha}\right). \quad (32) \]

Now subtract equation (30) from (29) we get,
\[ 2De^{-\alpha L} = Ee^{i\beta L}\left(1 - i\frac{\beta}{\alpha}\right) \]
\[ D = \frac{Ee^{i\beta L}}{2e^{-\alpha L}}\left(1 - i\frac{\beta}{\alpha}\right). \quad (33) \]
Inserting values of $C$ and $D$ from equations (32) and (33) into equation (31) results in,

$$A = \frac{1}{2} \left( \frac{E e^{i\beta L}}{2e^{\alpha L}} \right) \left( 1 + i \frac{\beta}{\alpha} \right) \left( 1 - i \frac{\alpha}{k} \right) + \frac{1}{2} \left( \frac{E e^{i\beta L}}{2e^{-\alpha L}} \right) \left( 1 - i \frac{\beta}{\alpha} \right) \left( 1 + i \frac{\alpha}{k} \right)$$

$$= \frac{E e^{i\beta L}}{4} \left[ e^{-\alpha L} \left( 1 + i \frac{\beta}{\alpha} - i \frac{\alpha}{k} + \beta \right) + e^{\alpha L} \left( 1 + i \frac{\beta}{\alpha} - i \frac{\alpha}{k} + \beta \right) \right]$$

$$= \frac{E e^{i\beta L}}{4} e^{-\alpha L} \left\{ (1 + \frac{\beta}{k}) + i \left( \frac{\beta}{\alpha} - \frac{\alpha}{k} \right) \right\} + e^{\alpha L} \left\{ (1 + \frac{\beta}{k}) - i \left( \frac{\beta}{\alpha} - \frac{\alpha}{k} \right) \right\}$$

$$= \frac{E e^{i\beta L}}{4} \left[ e^{-\alpha L} \left( 1 + \frac{\beta}{k} \right) + ie^{-\alpha L} \left( \frac{\beta}{\alpha} - \frac{\alpha}{k} \right) + e^{\alpha L} \left( 1 + \frac{\beta}{k} \right) - ie^{\alpha L} \left( \frac{\beta}{\alpha} - \frac{\alpha}{k} \right) \right].$$

and after some rearrangement,

$$A = \frac{E e^{i\beta L}}{4} \left[ (e^{\alpha L} + e^{-\alpha L}) \left( 1 + \frac{\beta}{k} \right) - i( e^{\alpha L} - e^{-\alpha L} ) \left( \frac{\beta}{\alpha} - \frac{\alpha}{k} \right) \right]$$

$$= \frac{E e^{i\beta L}}{2} \left[ \frac{e^{\alpha L} + e^{-\alpha L}}{2} \right] \left( 1 + \frac{\beta}{k} \right) - i \left( \frac{e^{\alpha L} - e^{-\alpha L}}{2} \right) \left( \frac{\beta}{\alpha} - \frac{\alpha}{k} \right) \right].$$

Using $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$,

$$A = \frac{E e^{i\beta L}}{2} \left[ \cosh(\alpha L) \left( 1 + \frac{\beta}{k} \right) - i \sinh(\alpha L) \left( \frac{\beta}{\alpha} - \frac{\alpha}{k} \right) \right]. \quad (34)$$

Taking the complex conjugate of equation (34),

$$A^* = \frac{E^* e^{-i\beta L}}{2} \left[ \cosh(\alpha L) \left( 1 + \frac{\beta}{k} \right) + i \sinh(\alpha L) \left( \frac{\beta}{\alpha} - \frac{\alpha}{k} \right) \right]. \quad (35)$$

Multiply equations (34) and (35) yields,

$$A^* A = \frac{E e^{i\beta L}}{2} \left[ \cosh(\alpha L) \left( 1 + \frac{\beta}{k} \right) - i \sinh(\alpha L) \left( \frac{\beta}{\alpha} - \frac{\alpha}{k} \right) \right] \cdot$$

$$\frac{E^* e^{-i\beta L}}{2} \left[ \cosh(\alpha L) \left( 1 + \frac{\beta}{k} \right) + i \sinh(\alpha L) \left( \frac{\beta}{\alpha} - \frac{\alpha}{k} \right) \right]$$

$$|A|^2 = \frac{|E|^2}{4} \left[ \cosh^2(\alpha L) \left( 1 + \frac{\beta}{k} \right)^2 + \sinh^2(\alpha L) \left( \frac{\beta}{\alpha} - \frac{\alpha}{k} \right)^2 \right]$$

$$|A|^2 = \frac{1}{4} \left[ (1 + \sinh^2(\alpha L)) \left( 1 + \frac{\beta}{k} \right)^2 + \sinh^2(\alpha L) \left( \frac{\beta}{\alpha} - \frac{\alpha}{k} \right)^2 \right]$$

$$= \frac{1}{4} \left[ \left( 1 + \frac{\beta}{k} \right)^2 + \sinh^2(\alpha L) \left\{ \left( 1 + \frac{\beta}{k} \right)^2 + \left( \frac{\beta}{\alpha} - \frac{\alpha}{k} \right)^2 \right\} \right]$$

$$= \frac{1}{T}.$$
obtain. Substitute values of \(k, \alpha\) and \(\beta\) we get,

\[
\frac{|A|^2}{|E|^2} = T^{-1} = \frac{1}{4}\left[ (1 + \frac{\sqrt{2m(E-W_0)}}{\frac{\hbar}{\sqrt{2mE}}})^2 \right. \\
+ \left. \frac{\sqrt{2m(E-W_0)}}{\frac{\hbar}{\sqrt{2mE}}} - \frac{\sqrt{2m(V_0-E)}}{\frac{\hbar}{\sqrt{2mE}}} \right]^2 \\
= \frac{1}{4}\left[ (1 + \frac{E-W_0}{E})^2 + \sinh^2\left(\frac{L}{\hbar}\sqrt{2m(V_0-E)}\right) \right. \\
+ \left. \left(\frac{E-W_0}{V_0-E} - \frac{V_0-E}{E}\right)^2 \right].
\]

(g) If the barrier acts like a 50:50 beamsplitter, the transmission probability will be 0.5.

Setting \(T = 0.5\), in the last expression of part (e),

\[
\left[ 1 + \frac{V_0^2}{4E(V_0-E)} \sinh^2\left(\frac{L}{\hbar}\sqrt{2m(V_0-E)}\right) \right]^{-1} = \frac{1}{2} \\
1 + \frac{V_0^2}{4E(V_0-E)} \sinh^2\left(\frac{L}{\hbar}\sqrt{2m(V_0-E)}\right) = 2 \\
\frac{V_0^2}{4E(V_0-E)} \sinh^2\left(\frac{L}{\hbar}\sqrt{2m(V_0-E)}\right) = 1 \\
\frac{4E(V_0-E)}{V_0^2} = \sinh^2\left(\frac{L}{\hbar}\sqrt{2m(V_0-E)}\right) \\
\frac{\sqrt{4E(V_0-E)}}{V_0} = \sinh\left(\frac{L}{\hbar}\sqrt{2m(V_0-E)}\right) \\
\frac{2\sqrt{E\varepsilon}}{V_0} = \sinh\left(\frac{L}{\hbar}\sqrt{2m\varepsilon}\right),
\]

where \(\varepsilon = V_0 - E\). The above expression is the required relation between \(E, V_0\) and \(L\).

**Answer 17:**

From the figure we can see that the wavenumber for region \(0 \leq x \leq L/2\) is,

\[
k = \frac{\sqrt{2mE}}{\hbar},
\]

while the wavenumber for region \(L/2 \leq x \leq L\) is,

\[
k' = \frac{\sqrt{2m(E-V_0)}}{\hbar}.
\]

Since \(k' < k, \lambda > \lambda'\), option (d) is the correct answer. The wavelength is longer in the right half of the potential well.
Answer 18:
Option (d) is the correct answer. From the figure, we observe that in region I the wavefunction is,

\[
k_I^2 = \frac{2mE}{\hbar^2} = \frac{2m(2V_0)}{\hbar^2} = \frac{4mV_0}{\hbar^2},
\]

and likewise for region II,

\[
k_{II}^2 = \frac{2mE}{\hbar^2} = \frac{2m(V_0)}{\hbar^2} = \frac{2mV_0}{\hbar^2}.
\]

Hence,

\[
\frac{k_I}{k_{II}} = \sqrt{2}.
\]

Since \( k = p/\hbar = mv/\hbar \), we have

\[
v_I = \sqrt{2}v_{II}.
\]

Answer 19:
In the absence of electric field,

\[
\lambda = \frac{h}{p} = \frac{h}{\sqrt{2mE}}, \text{ and speed is } v.
\]

In the presence of electric field, total energy of the electron changes to \((E - eu(x))\), resulting in the modified wavelength,

\[
\lambda' = \frac{h}{p'} = \frac{h}{\sqrt{2m(E - eu(x))}}, \text{ and the changed speed is } v'.
\]

Ratio between the momentums and speed is,

\[
\frac{p}{p'} = \frac{mv}{mv'} = \frac{v}{v'} = \sqrt{\frac{2mE}{2m(E - eu(x))}} = \sqrt{\frac{E}{(E - eu(x))}}.
\]

Hence (c) is the correct answer.

Answer 20:
The correct option is (a). The probability density \( \Psi^*(x, t)\Psi(x, t) = e^{-i(kx-\omega t)}e^{i(kx-\omega t)} = 1 \) is independent of position. So the particle can be found, with equal likelihood, anywhere along the \( x \) axis.
Answer 21:
The energy of electron is \( n^2 \frac{\pi^2 \hbar^2}{2mL^2} \) and in the ground state \( E_1 = \frac{\pi^2 \hbar^2}{2mL^2} \). If a photon shines on the quantum dot, the electron can be excited to the level, \( E_{\text{photon}} + \frac{\pi^2 \hbar^2}{2mL^2} \). If the electron is to remain confined,

\[
E_{\text{photon}} + \frac{\pi^2 \hbar^2}{2mL^2} < V_0 \\
\frac{\pi^2 \hbar^2}{2mL^2} < V_0 - E_{\text{photon}} \\
\frac{2mL^2}{\pi^2 \hbar^2} > \frac{1}{V_0 - E_{\text{photon}}} \\
L > \sqrt{\frac{\pi^2 \hbar^2}{2m(V_0 - E_{\text{photon}})}},
\]

which is the minimum diameter that the quantum dot must have.

Answer 22:
We are given that,

\[
\psi(x) = \frac{1}{L^2 + (x - x_o)^2/\alpha^2}.
\]

At \( x = x_0 \),

\[
\psi(x = x_0) = \frac{1}{L^2}.
\]

We can calculate \( (x - x_0) \) where the amplitude is half of maximum, i.e., \( \psi(x) = \frac{1}{2L^2} \).

\[
\Rightarrow \frac{1}{2L^2} = \frac{1}{L^2 + (x - x_o)^2/\alpha^2} \\
2L^2 = L^2 + \frac{(x - x_o)^2}{\alpha^2} \\
L^2 = \frac{(x - x_o)^2}{\alpha^2} \\
\alpha^2 L^2 = (x - x_o)^2 \\
\alpha L = (x - x_o) \\
\Delta x \sim 2\alpha L.
\]

According to uncertainty principle,

\[
\Delta p \geq \frac{\hbar}{\Delta x} = \frac{\hbar}{2\alpha L}.
\]
Since we are talking about estimates, option (a) is the correct answer.

Answer 23:

(a)

Fig. (a) shows the energy landscape before a voltage $V_0$ is applied. If the tip is given a positive potential, it’s potential energy, as seen by the electrons decreases. This results in a downwards bending of the energy profile.

![Fig. (c): Tunneling into metal](image1)

Now Fig. (c) and (d) show two scenarios; (c) is when $V_0$ is small such that an electron in the bulk metal can directly tunnel into the metal tip while keeping its energy constant. The thick arrow shows a tunneling electron. Whereas in (d), the $V_0$ is large, resulting in the excessive sloping of the potential energy — to such a large extent, that $E_{F_{(bulk)}}$ is higher in energy than $E_{F_{(tip)}} + \phi$. This would result in the electron being tunneled into air, instead of the metal.

(b) For tunneling into the metal tip, the required condition is evident from Fig. (c),

$$E_{F_{(bulk)}} \leq E_{F_{(tip)}} + \phi.$$  \hspace{1cm} (36)

We observe that,

$$E_{F_{(bulk)}} + \phi = E_{F_{(tip)}} + \phi + eV_0$$

$$\Rightarrow E_{F_{(bulk)}} = E_{F_{(tip)}} + eV_0.$$  \hspace{1cm} (37)

Comparing equations (42) and (37), we find out $eV_0 < \phi$ is the condition for tunneling into the metal tip. Likewise from Fig. (d), it is readily observable that $eV_0 > \phi$ is the condition for tunneling into air.
(c) For tunneling into the metal tip, $eV_0 < \phi$. To find the threshold voltage we set,

$$V_0 = \frac{\phi}{e} = 4 \text{ V}.$$  

Hence maximum electric field ensuring tunneling into the tip is,

$$E = \frac{V_0}{d} = \frac{4}{2 \times 10^{-9}} = 2 \times 10^9 \text{ V/m}.$$  

**Answer 24:**

For a half infinite well potential, the energy $E$ inside the well is directly proportional to $k^2$, $(E = \frac{\hbar^2 k^2}{2m})$ and the wave number $k$ is $k = \frac{2\pi}{\lambda}$. Therefore the maximum wavelength inside the well corresponds to the ground state energy. If the ground state energy is less than $V_0$, we will, for sure, have at least one bound state. The maximum possible wavelength inside the well is simply equal to $2a$, such that,

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{2a} = \frac{\pi}{a}.$$  

The energy for this longest wavelength inside the well is,

$$E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{2ma^2} = \frac{\hbar^2}{8\pi^2 ma^2}.$$  

For the existence of bound states it is necessary that $V_0 > E$. It implies that there will be no bound states if,

$$V_0 < \frac{\hbar^2}{8ma^2},$$  

which is the required result.

**Answer 25:**

\[ E > V_0 \]

\[ \begin{array}{ccc}
\text{I} & \text{II} & V_0 \\
\end{array} \]

\[ x=0 \]

Date: 21 April, 2014
The wavefunctions in region I, \( \psi_I(x) = Ae^{ik_1x} + Be^{-ik_1x} \), with \( k_1^2 = \frac{2mE}{\hbar^2} \), and

The wavefunctions in region II, \( \psi_{II}(x) = Ce^{ik_2x} \), with \( k_2^2 = \frac{2m(E - V_0)}{\hbar^2} \).

At \( x = 0 \),

\[
A + B = C \quad (38)
\]

and \( ik_1(A - B) = ik_2C \),

which are derived from continuity of wavefunction and its derivative at the discontinuity \( (x = 0) \).

From equation (38),

\[
\frac{k_1}{k_2}(A - B) = C \quad (40)
\]

Comparing equation (38) and (40),

\[
A + B = \frac{k_1}{k_2}(A - B)
\]

\[
A \left( 1 - \frac{k_1}{k_2} \right) = -B \left( 1 + \frac{k_1}{k_2} \right)
\]

\[
\frac{B}{A} = -\left( \frac{1 - (k_1/k_2)}{1 + (k_1/k_2)} \right) = -\left( \frac{k_2 - k_1}{k_2 + k_1} \right)
\]

The reflection coefficient is given by \( \frac{|B|^2}{|A|^2} \),

\[
\frac{|B|^2}{|A|^2} = \left( \frac{(k_2 - k_1)}{(k_2 + k_1)} \right)^2 = \left( \frac{k_2 - k_1}{k_2 + k_1} \right)^2.
\]

Setting this to 0.5, we have,

\[
\frac{(k_2 - k_1)}{(k_2 + k_1)} = 0.5
\]

\[
\frac{k_2 - k_1}{k_2 + k_1} = \pm \sqrt{0.5} \quad (41)
\]

At first consider

\[
\frac{k_2 - k_1}{k_2 + k_1} = +\sqrt{0.5}
\]

\[
k_2 - k_1 = \sqrt{0.5} (k_2 + k_1)
\]

\[
k_2(1 - \sqrt{0.5}) = k_1(1 + \sqrt{0.5})
\]

\[
\frac{k_2}{k_1} = \frac{1 + \sqrt{0.5}}{1 - \sqrt{0.5}} \approx 5.82
\]
which is the required ratio. Note that since $k_2 > k_1$, for 50% reflection (or 50% transmission), the discontinuity must in fact be a depression instead of an elevation, as shown below.

If instead, we take the other possibility, i.e. the R.H.S in equation (41) is with a negative sign,

$$\frac{k_2 - k_1}{k_2 + k_1} = -\sqrt{0.5}$$
$$k_2 - k_1 = -\sqrt{0.5} (k_2 + k_1)$$
$$k_2(1 + \sqrt{0.5}) = k_1(1 - \sqrt{0.5})$$
$$\frac{k_2}{k_1} = \frac{1 - \sqrt{0.5}}{1 + \sqrt{0.5}} = \frac{0.293}{1.707} = 0.17 .$$

In this second case, $k_2 < k_1$, so the discontinuity is an elevation, as shown below.

**Answer 26:**
In thermionic emission, heating increases the thermal energy of the electrons. These electrons are raised from the filled to the unfilled levels. Some of these excited electrons obtain enough energy to overcome the work function and can therefore be ejected into air.

**Answer 27:**
Option (d) is the correct answer. The quantum dot is given a variable positive potential $V_0$. An electron added to the quantum dot raises its coulomb energy by $e^2/2C_{\text{dot}}$. Hence
if energy is to be conserved and the electron transfer to the dot is to be favored, the starting energy of the dot must be lower by $e^2/2C_{dot}$, so that pickup of an extra electron is energetically permissible. If $V_0 = 0$, the electron cannot tunnel into $n = 7$ as it will raise the overall energy of the dot. If, however, $V_0 = e/2C_{dot}$, $n = 7$ is lowered in energy by $e^2/2C_{dot}$ and electron tunneling to $n = 7$ becomes energetically permissible. The electron tunnel! While keeping its total energy constant. The quantum dot is a receptacle lowering its energy in anticipation of an incoming electron, which raises the energy back to the original. Hence four peaks corresponding to tunneling to $n = 7, 8, 9, 10$ are observed.

**Answer 28:**

Option (c) is the correct answer. If $v$ is to go up in region III, $k$ must increase. Since $k \propto \sqrt{E-V}$, the difference between $E$ and $V$ must be higher, which is the situation in Fig.(c).

**Answer 29:**

In the refracting medium, the electric potential is positive and hence the potential energy seen by the electron, $V$, is lower. This means that the difference $E-V$ is larger, $k$ is larger, and hence the speed is slower, $v_2 < v_1$. Hence from Bethe’s law, $\sin \alpha < \sin \beta$, $\alpha < \beta$. The electron beam bends away from the normal. Hence option (c) is the correct answer.

**Answer 30:**

As we know that when a quantum object encounters a wider barrier, the tunneling transmission probability is lower. If the barrier is thinner the tunneling probability is higher. In the given figure we can see that the barrier on the right is thinner. Therefore it is more probable to find the electron in region $C$ as compared to $A$. Hence option (b) is the correct answer.

**Answer 31:**

One need to think carefully about this. Consider the accompanying figure.
The potential depression $V_0$ is large. Let’s find the reflection probability $R$. In region I,

$$\psi_I(x) = Ae^{ik_1x} + Be^{-ik_1x},$$

and for region II,

$$\psi_{II}(x) = Ce^{ik_2x},$$

where $k_1^2 = \frac{2mE}{h}$, and $k_2 = \frac{2m(E+V_0)}{h}$. At the point of the precipice, $x = 0$, $\psi_I(0) = \psi_{II}(0)$ and $\psi_I'(0) = \psi_{II}'(0)$. So $A + B = C$ and $ik_1(A - B) = ik_2C$. Eliminating $C$ from these equations,

$$A + B = \frac{ik_1(A - B)}{ik_2}$$

$$= \frac{k_1}{k_2} A - \frac{k_1}{k_2} B$$

$$B \left(1 + \frac{k_1}{k_2}\right) = A \left(\frac{k_1}{k_2} - 1\right)$$

$$B = \frac{k_1 - k_2}{k_1 + k_2}$$

$$A = \frac{k_1 + k_2}{k_1 + k_2}$$

$$R = \frac{|B|^2 k_1}{|A|^2 k_1}$$

$$= \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2.$$

If $V_0$ is very large, $k_2 \gg k_1$, $R$ becomes $\left(\frac{k_1}{k_2}\right)^2 = 1$. Since $R \simeq 1$, $T = 0$. There is zero probability for the particle to “fall over the edge” and enter region II. Hence option (a) is the correct answer.

**Answer 32:**
We are given that,
\[ E_{n_x, n_y, n_z} = \frac{\pi^2 \hbar^2}{2m} \left( \frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2} \right) \]
\[ = \frac{\pi^2 \hbar^2}{2m} (n_x^2 + n_y^2 + n_z^2) \quad \text{since} \quad a = b = c. \]

For ground state \((n_x, n_y, n_z) = (1, 1, 1)\), energy of the ground state will be,
\[ E(1, 1, 1) = \frac{\pi^2 \hbar^2}{2m} (1^2 + 1^2 + 1^2) \]
\[ = \frac{3 \pi^2 \hbar^2}{2m}. \]

Similarly for first excited state, \((n_x, n_y, n_z) = (2, 1, 1)\), energy of the first excited state will be,
\[ E(2, 1, 1) = \frac{\pi^2 \hbar^2}{2m} (2^2 + 1^2 + 1^2) \]
\[ = \frac{6 \pi^2 \hbar^2}{2m}. \]

Energy difference for these two energy levels is,
\[ E(2, 1, 1) - E(1, 1, 1) = \frac{6 \pi^2 \hbar^2}{2m} - \frac{3 \pi^2 \hbar^2}{2m} \]
\[ = \frac{3 \pi^2 \hbar^2}{2m}. \]

Hence option (c) is the correct answer.

**Answer 33:**
As we know that general equation of wave function is,
\[ \Psi(x, t) = A e^{i k x - \omega t}. \]

Comparison of this equation with the given wave equation of for the free particle yields,
\[ k = 2.5 \times 10^{11} \text{ m}^{-1} \]
\[ \omega = 2.1 \times 10^{13} \text{ s}^{-1}. \]

Mass of the particle can be calculated by the dispersion relation,
\[ \omega^2 = \frac{k}{m} \]
\[ \Rightarrow m = \frac{k}{\omega^2} \]
\[ = \frac{2.5 \times 10^{11}}{(2.1 \times 10^{13})^2} \]
\[ = 5.7 \times 10^{-16} \text{ kg}. \]
Hence option (d) is the correct answer.

You don’t really need to remember the dispersion relationship. Look at the TDSE in the absence of $V$:

$$\text{TDSE} : -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} \Psi(x,t) = i\hbar \frac{d}{dt} \Psi(x,t).$$

Inserting the supplied wave function into the above, the relationship $\omega^2 = k/m$ automatically pops out. Students are tempted to use $E = h\omega$ and $E = p^2/2m$. The former relationship does not hold for all particles, it is specific in its meaning—it says that energy of a photon $E$ is related to the frequency of the electromagnetic wave associated with the photon. Blindly using these relations is wrong!

**Answer 34:**

Since the length of infinite well is very large i.e. 30 cm, for a small amount of energy 1 eV, the number of nodes will be very large i.e. $4.9 \times 10^8$. Since the number of nodes is very large, the waves are “squeezed” close together, the de Broglie wavelength is extremely small obscuring chances of observing the quantum wave behavior at the classical macroscopic scale. At such a high value of $n$, quantum effects are not visible. Another way of looking at this is that the wave function is such that the probability of finding the electron becomes equal everywhere, i.e. it imparting the electron a continuous quality rather than quantized. All of this ties in well with Bohr’s corresponding principle.

**Answer 35:**

We are given that,

$$\begin{align*}
\text{At } t = 0 & \quad \Psi(x,0) = \frac{1}{\sqrt{5L}} \left( \psi_1(x) + 2\psi_2(x) \right) \\
\text{At any time } t & \quad \Psi(x,t) = \frac{1}{\sqrt{5L}} \left( \psi_1(x)e^{-iE_1t/\hbar} + 2\psi_2(x)e^{-iE_2t/\hbar} \right).
\end{align*}$$

Date: 21 April, 2014
If $E_1$ is the energy of ground state, energy of first excited state will be $E_2 = 4E_1$.

$$\Rightarrow \Psi(x, t) = \frac{1}{\sqrt{5L}} \left( \psi_1(x)e^{-iE_1t/h} + 2\psi_2(x)e^{-i4E_1t/h} \right)$$

At $t = \frac{\pi h}{E_1}$

$$\Psi(x, t) = \frac{1}{\sqrt{5L}} \left( \psi_1(x)e^{-i\pi} + 2\psi_2(x)e^{-i4\pi} \right)$$

$$= \frac{1}{\sqrt{5L}} \left( \psi_1(x)(-1) + 2\psi_2(x)(+1) \right)$$

$$= \frac{1}{\sqrt{5L}} \left( -\psi_1(x) + 2\psi_2(x) \right).$$

Now we need to find out what $-\psi_1(x) + 2\psi_2(x)$ looks like. The construction is seen in the series of diagrams below.

Hence option (a) is the correct answer.

**Answer 36:**

Option (d) is the correct answer. The wavefunction is zero at $x \geq b$ because of the infinite potential and extends into the region $x \leq a$. Furthermore, the value of $k$ increases and wavelength decreases as we go from $x = a$ to $x = b$.

**Answer 37:**

Option (a) is the correct answer.
Answer 38:

We are given the wavefunction of free particle,
\[
\Psi(x, t) = A(e^{ikx} + e^{-ikx})e^{-i\frac{Et}{\hbar}}
\]
\[
\Rightarrow \Psi^* (x, t) = A^* (e^{-ikx} + e^{ikx})e^{i\frac{Et}{\hbar}}
\]
\[
\Psi^* (x, t) \Psi (x, t) = A^* (e^{-ikx} + e^{ikx}) \cdot A(e^{ikx} + e^{-ikx})e^{-i\frac{Et}{\hbar}} e^{i\frac{Et}{\hbar}}
\]
\[
= A^2 (1 + e^{-2ikx} + e^{2ikx} + 1)
\]
\[
= A^2 (2 + e^{-2ikx} + e^{2ikx}).
\]

Using \( \cos x = \frac{e^{ix} + e^{-ix}}{2} \)
\[
|\Psi^2(x, t)|^2 = A^2 (2 + 2 \cos(2kx))
\]
\[
= A^2 (2 + 2 \cos(2kx))
\]
\[
= 2A^2 (1 + \cos(2kx))
\]
\[
= 2A^2 \cdot 2 \cos^2(kx)
\]
\[
= 4A^2 \cdot \cos^2(kx)
\]

1. True since \( p(x) = |\Psi^2(x, t)|^2 \) is independent of time.

2. False since \( p(x) \) depends on \( x \) and changes with \( x \).

3. True because the forward and backward propagating waves have equal amplitudes and the probability density does not change with time.

Answer 39:

According to uncertainty principle,
\[
\Delta p \Delta x \geq \frac{\hbar}{2}
\]

If particle moves in a circle of radius \( r \) and angular momentum \( L \), then
\[
L = pr
\]
\[
\Rightarrow \Delta L = \Delta pr
\]
\[
\Rightarrow \Delta p = \frac{\Delta L}{r}
\]
and \( \Delta x = r \Delta \theta \).
Using these values of $\Delta p$ and $\Delta x$ in uncertainty relation,

$$\frac{\Delta L \cdot r \Delta \theta}{r} \geq \frac{\hbar}{2}$$

$$\Delta L \Delta \theta \geq \frac{\hbar}{2}.$$ 

Hence option (a) is the correct answer.

**Answer 40:**

We are given that,

$$\text{Tip-sample distance} = \alpha = 1 \text{ nm}^{-1} = 1 \times 10^9 \text{ m}^{-1}$$

$$\text{Distance covered by tip} = \Delta L = 0.1 \text{ nm} = 0.1 \times 10^{-9} \text{ m}.$$ 

Tunneling probability is,

$$T_i = e^{-2\alpha L}.$$ 

If tip moves closer to the surface by $\Delta L$, final tunneling probability will become,

$$T_f = e^{-2\alpha (L-\Delta L)}.$$ 

Ratio of tunneling probabilities is,

$$\frac{T_f}{T_i} = \frac{e^{-2\alpha (L-\Delta L)}}{e^{-2\alpha L}} = e^{2\alpha \Delta L} = e^{2 \times 10^9 \times 0.1 \times 10^{-9}} = e^{0.2} = 1.22.$$ 

Hence there is a 22% increase in the tunneling current and the correct answer is (b).

**Answer 41:**

(a) We are given that,

$$E = pc.$$
Uncertainty in energy is,

\[ \Delta E \approx c \Delta p. \tag{42} \]

Uncertainty in momentum \( \Delta p \) for a photon confined to the tiny box of size \( \Delta x \) can be calculated by using the uncertainty relation,

\[ \Delta x \Delta p \geq \frac{\hbar}{2}, \]

where \( \hbar \) is the reduced Planck's constant.

\[ \Rightarrow \Delta p \geq \frac{\hbar}{2\Delta x}. \]

Using this value of \( \Delta p \) in equation (42) yields,

\[ \Delta E \sim c \left( \frac{\hbar}{2\Delta x} \right) \]
\[ \Delta E \sim \frac{c\hbar}{2\Delta x} \]

(b) We are given that,

\[ \Delta E \sim E \]
\[ \Rightarrow E = \frac{c\hbar}{2\Delta x} \]

Effective mass can be calculated by using energy-mass relationship,

\[ E = m_{\text{eff}}c^2, \]

where \( m_{\text{eff}} \) is the effective mass of the photon.

\[ \Rightarrow m_{\text{eff}} = \frac{E}{c^2} \]
\[ = \frac{c\hbar}{2\Delta x} \cdot \frac{1}{c^2} \]
\[ = \frac{\hbar}{2c\Delta x}. \]

(c) We are given that,

Mass of the star = \( M \)
Radius of the star = \( R \)
Let mass of the object = \( m \),
and let speed of the object to escape from the star’s gravitational pull is $v_{\text{esc}}$. In order to escape from the gravitational pull of star, kinetic energy of the object should be at least equal to the gravitational potential energy.

$$|\text{K.E.}| = |\text{P.E.}|$$

$$\Rightarrow \frac{1}{2}mv_{\text{esc}}^2 = \frac{GmM}{R}$$

$$v_{\text{esc}}^2 = \frac{2GM}{R}$$

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}}.$$  

Which is the required result.

**d)** Using the relation of $v_{\text{esc}}$ derived in part (c)

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}},$$

where $v_{\text{esc}} = c$,

$$c = \sqrt{\frac{2GM}{R}}$$

$$c^2 = \frac{2GM}{R}.$$

Setting $R = \Delta x$ and $M = m_{\text{eff}},$

$$c^2 = \frac{2Gm_{\text{eff}}}{\Delta x} = \frac{2G}{\Delta x} \cdot \frac{h}{2c\Delta x} = \frac{Gh}{c^2}$$

$$\Delta x^2 = \frac{Gh}{c^3}$$

$$\Delta x = \sqrt{\frac{Gh}{c^3}}$$

**e)**

$$\Delta x = \sqrt{\frac{Gh}{c^3}}$$

$$= \sqrt{\frac{6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2 \times 1.06 \times 10^{-34} \text{ Js}}{(3.0 \times 10^8 \text{ ms}^{-1})^3}}$$

$$= \sqrt{2.62 \times 10^{-70} \text{ m}^2}$$

$$= 1.6 \times 10^{-35} \text{ m.}$$
(f) We are given that,

\[
\text{Diameter of photon} = 2 \text{ fm} = 2 \times 10^{-15} \text{ m}
\]

\[
\text{Diameter of photon in Planck's length} = \frac{2 \times 10^{-15} \text{ m}}{1.6 \times 10^{-35} \text{ m}/\text{Planck length}} = 1.24 \times 10^{20} \text{ Planck lengths}.
\]

**Answer 42:**

We are given that,

Barrier length = \( L = 35 \text{ fm} = 35 \times 10^{-15} \text{ m} \)

Barrier height = \( (V_0 - E) = 5 \text{ MeV} = 5 \times 10^6 \times 1.6 \times 10^{-19} \text{ J} \)

\[
= 8 \times 10^{-13} \text{ J}
\]

Tunneling probability = \( T = e^{-2\alpha L} \)

where

\[
\alpha = \frac{\sqrt{2m(V_0 - E)}}{\hbar}
\]

and mass of the nucleus = \( m = 4 \times 1.67 \times 10^{-27} \text{ kg} = 6.68 \times 10^{-27} \text{ kg} \).

In order to calculate probability of escape \( T \), let’s first calculate \( \alpha \).

\[
\alpha = \frac{\sqrt{2m(V_0 - E)}}{\hbar} = \frac{\sqrt{2 \times 6.68 \times 10^{-27} \text{ kg} \times 5 \times 10^6 \times 1.6 \times 10^{-19} \text{ J}}}{1.06 \times 10^{-34} \text{ Js}} = 9.75 \times 10^{14} \text{ m}^{-1}.
\]

Tunneling probability is,

\[
T = e^{-2\alpha L} = e^{-2 \times 9.75 \times 10^{14} \text{ m}^{-1} \times 35 \times 10^{-15} \text{ m}} = e^{-68.25} = 2.29 \times 10^{-30}.
\]
The probability that an alpha particle hitting the nucleus surface will escape is $2.29 \times 10^{-30}$.

If no. of hits per second $= 5 \times 10^{21}/s$

\[
\left( \frac{\text{Probability of escape}}{\text{in a day}} \right) = T = 2.29 \times 10^{-30} \times 5 \times 10^{21}/s \times 24 \times 60 \times 60 \text{ s}
\]

$= 9.89 \times 10^{-4}$

$= 9.89 \times 10^{-2} \%$