Final Exam 2013: Modern Physics
Solution

Name: 

Roll no:

• Write your name and roll number in the space specified above.

• This exam comprises two parts, A and B. Part A comprises 23 questions. The most appropriate answer is to be circled on the question paper. Part A is to be filled on the question paper and returned.

• For answering part B, you will use the provided blue answer books. There are two questions in part B.

• Part A contains 15 pages including this page.

• Part B contains 2 pages.
Fundamental constants and other useful information

\[ c = \text{Speed of light} = 3.0 \times 10^8 \text{ ms}^{-1} \]
\[ h = \text{Planck's constant} = 6.63 \times 10^{-34} \text{ Js} \]
\[ \hbar = \text{Reduced Planck's constant} = \frac{\hbar}{2\pi} = 1.06 \times 10^{-34} \text{ Js} \]
\[ k_B = \text{Boltzmann constant} = 1.38 \times 10^{-23} \text{ J/K} \]
\[ R = \text{Rydberg constant} = 1.1 \times 10^7 \text{ m}^{-1} \]
\[ m_e = \text{Mass of electron} = 9.11 \times 10^{-31} \text{ kg} \]
\[ m_p = \text{Mass of proton} = 1.67 \times 10^{-27} \text{ kg} \]
\[ 1\text{eV} = 1.6 \times 10^{-19} \text{ J} \]
\[ g = \text{Acceleration due to gravity} = 10.0 \text{ m s}^{-2} \]
\[ G = \text{Gravitational constant} = 6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2 \]
\[ \frac{1}{4\pi\varepsilon_0} = \text{Coulomb's constant} = 9 \times 10^9 \text{ N m}^2/\text{C}^2 \]

TISE: \[ -\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x) \]

TDSE: \[ -\frac{\hbar^2}{2m} \frac{d^2 \Psi(x, t)}{dx^2} + V(x)\Psi(x, t) = i\hbar \frac{d}{dt} \Psi(x, t) \]
PART A

Attempt all questions. Mark your answers on these sheets and return. All MCQ’s are three marks each.

1. A metal is held at zero voltage. The energy diagram at the metal-air interface is shown.

In thermionic emission, electrons are ejected from the metal surface because:

(a) The work function $\phi$ increases.

(b) The work function $\phi$ decreases.

(c) The potential energy seen by the electrons in the air slopes downward.

(d) Increasing temperature makes more electrons jump into unfilled levels increasing the fraction of electrons with thermal energy beyond $\phi$.

(e) The Fermi level $E_F$ decreases.

**Answer 1:**

In thermionic emission, heating increases the thermal energy of the electrons. These electrons are raised from the filled to the unfilled levels. Some of these excited electrons obtain enough energy to overcome the work function and can therefore be ejected into air.
The figure shows the energy diagram for a metal in which electrons fill energy levels upto \( E_F \). A thin insulating oxide layer separates the metal from a quantum dot with only ten quantized energy levels. The quantum dot is given a positive potential \( V_0 \) with respect to the metal, enabling an electron to tunnel across the oxide layer. Which one of these plots shows the correct behavior of the tunneling current \( i \) from metal to the quantum dot. At \( V_0 = 0 \), \( E_F \) is at the same energy as the \( n = 7 \) quantum level.

(a) \[ i \] \( \rightarrow V_s \)

(b) \[ i \] \( \rightarrow V_s \)

(c) \[ i \] \( \sim e / 2C_{\text{dot}} \) \( \rightarrow V_s \)

(d) \[ i \] \( \sim e / 2C_{\text{dot}} \) \( \rightarrow V_s \)

(e) \[ i \] \( \sim e / 2C_{\text{dot}} \) \( \rightarrow V_s \)
**Answer 2:**

Option (d) is the correct answer. The quantum dot is given a variable positive potential $V_0$. An electron added to the quantum dot raises its coulomb energy by $e^2/2C_{\text{dot}}$. Hence if energy is to be conserved and the electron transfer to the dot is to be favored, the starting energy of the dot must be **lower** by $e^2/2C_{\text{dot}}$, so that pickup of an extra electron is energetically permissible. If $V_0 = 0$, the electron cannot tunnel into $n = 7$ as it will raise the overall energy of the dot. If, however, $V_0 = e/2C_{\text{dot}}$, $n = 7$ is lowered in energy by $e^2/2C_{\text{dot}}$ and electron tunneling to $n = 7$ becomes energetically permissible. The electron tunnel! While keeping its total energy constant. The quantum dot is a receptacle lowering its energy in anticipation of an incoming electron, which raises the energy back to the original. Hence four peaks corresponding to tunneling to $n = 7, 8, 9, 10$ are observed.

3. An electron is injected into a potential energy landscape from the left region I as shown below. It encounters a potential step. The energy of the electron is $E$ and $E < |V_0|$. If the electron is to emerge in region III with a faster speed, the appropriate potential step is given by which of the following?

![Potential Energy Diagrams](image)

(e) The speed of the electron cannot increase.
**Answer 3:**

Option (c) is the correct answer. If \( v \) is to go up in region III, \( k \) must increase. Since \( k \propto \sqrt{E-V} \), the difference between \( E \) and \( V \) must be higher, which is the situation in Fig.(c).

4.

![Diagram of a pendulum]

A pendulum is a harmonic oscillator. It completes one round trip in 1 s. According to quantum physics, its minimum energy in Joules is,

(a) Zero. 
(b) \( 6.63 \times 10^{-34} \).
(c) I need to know the angle \( \theta \) to answer this question.
(d) \( 3.315 \times 10^{-34} \).
(e) It is negative.

**Answer 4:**

Minimum energy of a harmonic oscillator is,

\[
E_{\text{min.}} = \frac{1}{2} h\omega,
\]

where \( h = \hbar/2\pi \) and \( \omega = 2\pi/T \), where \( T = 1 \) s is the time period of harmonic oscillator. Therefore,

\[
E_{\text{min.}} = \frac{1}{2} \left( \frac{h}{2\pi} \right) \left( \frac{2\pi}{T} \right) = \frac{1}{2} \left( \frac{h}{T} \right) = \frac{1}{2} \left( \frac{6.63 \times 10^{-34} \text{ Js}}{1 \text{ s}} \right) = 3.315 \times 10^{-34} \text{ J}.
\]

Hence option (d) is the correct answer.
5. Suppose the maximum angle \( \theta \) of the pendulum discussed in the previous question is such that the bob goes to a maximum height of 1 mm. The mass of the pendulum is 1 gram and \( g = 10 \text{ ms}^{-2} \). The bob starts oscillating and eventually comes to rest, losing all of its energy. Energy is lost in the form of a phonon when the oscillator makes a transition from the \( p/\text{th} \) quantum level to \( (p-1)/\text{th} \) quantum level. How many phonons are emitted in the process?

(a) Negligible. (b) One. (c) \( \approx 1.5 \times 10^{28} \). (d) \( \approx 1.5 \times 10^{33} \). (e) None of the above.

**Answer 5:**

We are given that,

- Height of bob \( y = 1 \text{ mm} = 10^{-3} \text{ m} \)
- Mass of the pendulum \( m = 1 \text{ g} = 10^{-3} \text{ kg} \)
- Acceleration due to gravity \( g = 10 \text{ m/s}^2 \).

the maximum energy of the photon is,

\[
mgy = \left(n + \frac{1}{2}\right) h \omega = \left(n + \frac{1}{2}\right) \frac{h}{2\pi} \frac{2\pi}{T} = \left(n + \frac{1}{2}\right) \frac{h}{T},
\]

and the principle quantum number is,

\[
\Rightarrow n = \frac{mgy}{h} - \frac{1}{2} = \frac{10^{-3} \text{ kg} \times 10 \text{ m/s}^2 \times 10^{-3} \text{ m}}{6.63 \times 10^{-34} \text{ Js}} - \frac{1}{2} \approx 1.5 \times 10^{28}.
\]

As the pendulum loses all of its energy, the quantum number decrements by 1, i.e., from \( n \) to \( n-1 \) to \( n-2 \) to \( n-3 \) all the way to \( n = 0 \). On each decrement, phonon is emitted. Hence the total number of phonons emitted is \( \sim 1.5 \times 10^{28} \). Therefore option (c) is the correct answer.
6. Snell’s law of refraction determines the bending of light across an interface. For sure, electrons are also waves and can be refracted. The corresponding law for electrons is called Bethe’s law and is given by,

$$\frac{\sin \alpha}{\sin \beta} = \frac{v_2}{v_1},$$

where $\alpha$ is the angle of incidence measured from the normal to the interface, $\beta$ is the angle of refraction also measured from the normal, $v_1$ is the speed of electron in the incident medium and $v_2$ is the speed in the refracted medium. Now a beam of electrons is made to pass through two hollow cylinders with an applied voltage difference. Which of the following diagrams show the correct trajectory of electrons?
**Answer 6:**

In the refracting medium, the electric potential is positive and hence the potential energy seen by the electron, \( V \), is lower. This means that the difference \( E - V \) is larger, \( k \) is larger, and hence the speed is slower, \( v_2 < v_1 \). Hence from Bethe’s law, \( \sin \alpha < \sin \beta, \alpha < \beta \). The electron beam bends away from the normal. Hence option (c) is the correct answer.

7. A proton of rest mass \( m_0 \) is accelerated to half the speed of light. Its de Broglie wavelength is,
(a) $\sqrt{3}h/(m_0c)$. 
(b) $2h/(\sqrt{3}m_0c)$. 
(c) $4h/(\sqrt{3}m_0c)$. 
(d) $4h/(m_0c)$. 
(e) A proton does not have a wavelength.

**Answer 7:**

From the de Broglie hypothesis we know that wavelength associated with the matter wave is,

$$\lambda = \frac{h}{p} = \frac{h}{mv}, \quad \text{since} \quad p = mv,$$

where according to special relativity if $m_0$ is the rest mass of the proton,

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \Rightarrow \lambda = \frac{h}{m_0v}\sqrt{1 - \frac{v^2}{c^2}}.$$

Since we are given that, speed of proton $v = c/2$,

$$\lambda = \frac{2h}{m_0c}\sqrt{1 - \frac{c^2}{4c^2}} = \frac{2h}{m_0c} \frac{\sqrt{3}}{2} = \frac{\sqrt{3}h}{m_0c}.$$

Hence option (a) is the correct answer.

8. An electron starts off in the region $B$, trapped in a well. The potential energy $V(x)$ along position $x$ is shown.

![Diagram](attachment:diagram.png)
Now suppose some time lapses. From a quantum viewpoint, which of the regions $A$ or $C$, is the electron more probable to be found?

(a) Region $A$.

(b) Region $C$.

(c) Equal probability of being found in $A$ and $C$.

(d) The electron absolutely cannot leave region $B$.

(e) None of the above.

**Answer 8:**

As we know that when a quantum object encounters a wider barrier, the tunneling transmission probability is lower. If the barrier is thinner the tunneling probability is higher. In the given figure we can see that the barrier on the right is thinner. Therefore it is more probable to find the electron in region $C$ as compared to $A$. Hence option (b) is the correct answer.

9. A particle moving in a region of zero force encounters a precipice—a sudden drop in the potential energy to an extremely large negative value. What is the probability that it will “go over the edge”, i.e., it will enter the negative potential energy region?

(a) Almost zero.  
(b) Almost one.

(c) $\approx 1/2$.  
(d) $> 1/2$.

(e) None of the above.

**Answer 9:**

One need to think carefully about this. Consider the accompanying figure.

The potential depression $V_0$ is large. Let’s find the reflection probability $R$. In region I,
\[ \psi_I(x) = Ae^{ik_1x} + Be^{-ik_1x}, \]

and for region II,

\[ \psi_{II}(x) = Ce^{ik_2x}, \]

where \( k_1^2 = \frac{2mE}{\hbar^2} \), and \( k_2 = \frac{2m(E+V_0)}{\hbar} \). At the point of the precipice, \( x = 0 \), \( \psi_I(0) = \psi_{II}(0) \) and \( \psi'^I(0) = \psi'^{II}(0) \). So \( A + B = C \) and \( ik_1(A - B) = ik_2C \). Eliminating \( C \) from these equations,

\[ \frac{A + B}{i} = \frac{ik_1(A - B)}{ik_2} \]

\[ A = \frac{k_1}{k_2} \quad B = \frac{k_1 - k_2}{k_1 + k_2} \]

\[ R = \frac{|B|^2}{|A|^2} = \left( \frac{k_1 - k_2}{k_1 + k_2} \right)^2. \]

If \( V_0 \) is very large, \( k_2 \gg k_1 \), \( R \) becomes \( \left( \frac{k_1}{k_2} \right)^2 = 1 \). Since \( R \approx 1 \), \( T = 0 \). There is zero probability for the particle to “fall over the edge” and enter region II. Hence option (a) is the correct answer.

10. An LED (light emitting diode) emits light when electrons fall from a top set of levels (conduction band) to a bottom set of levels (called a valence band). These levels are separated by an energy gap \( E_g \). An electric current supplies the electrons.
A current of 2.5 mA flows through an LED with $E_g = 1.4$ eV. Assuming that each current-carrying electron drops into a hole, thereby emitting a single photon, what is the power emitted in the light?

(a) $5.6 \times 10^{-22}$ W.  
(b) 3.5 mW.  
(c) 2.5 mW.  
(d) 1.4 mW.  
(e) The power cannot be calculated using the provided information.

**Answer 10:**

we are given that,

Current through LED = $I = 2.5$ mA = $2.5 \times 10^{-3}$ A

Energy gap = $E_g = 1.4$ eV = $1.4 \times 1.6 \times 10^{-19}$ J

No. of electrons flowing per unit time = $\frac{I}{\text{Charge of electron}} = \frac{I}{e}$

No. of photons emitted per unit time = $\frac{I}{\text{Charge of proton}} = \frac{I}{e}$

Energy of each photon = $E_g$.

Hence power emitted in the light is,

$$ P = \frac{I}{e} \times E_g $$

$$ P = \frac{2.5 \times 10^{-3} \, \text{A}}{1.6 \times 10^{-19} \, \text{C}} \times 1.4 \times 1.6 \times 10^{-19} \, \text{J} $$

$$ = 3.5 \times 10^{-3} \, \text{J/s} $$

$$ = 3.5 \, \text{mW}. $$

Hence option (b) is the correct answer.

11. An electron is trapped inside a three-dimensional quantum dot. The energy is quantized in three dimensions according to,

$$ E_{n_x, n_y, n_z} = \frac{\pi^2 \hbar^2}{2m} \left( \frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2} \right), $$

where $a$, $b$ and $c$ are the confining dimensions of the box (= dot) and $n_x$, $n_y$, $n_z$ are the three quantum numbers, each one of them being a positive integer.
If $a = b = c$, the energy difference between the ground and the first excited state is,

(a) $\frac{\pi^2h^2}{2ma^2}$.

(b) $9\frac{\pi^2h^2}{2ma^2}$.

(c) $3\frac{\pi^2h^2}{2ma^2}$.

(d) $\frac{\pi^2h^2}{ma^2}$.

(e) There are more than one “first excited states” all with different energies. Hence this question cannot be answered.

**Answer 11:**

We are given that,

$$E_{n_x,n_y,n_z} = \frac{\pi^2h^2}{2m} \left( \frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2} \right)$$

$$= \frac{\pi^2h^2}{2m} (n_x^2 + n_y^2 + n_z^2) \quad \text{since} \quad a = b = c.$$

For ground state $(n_x, n_y, n_z) = (1, 1, 1)$, energy of the ground state will be,

$$E(1, 1, 1) = \frac{\pi^2h^2}{2m} (1^2 + 1^2 + 1^2)$$

$$= 3\frac{\pi^2h^2}{2m}.$$

Similarly for first excited state, $(n_x, n_y, n_z) = (2, 1, 1)$, energy of the first excited state will be,

$$E(2, 1, 1) = \frac{\pi^2h^2}{2m} (2^2 + 1^2 + 1^2)$$

$$= 6\frac{\pi^2h^2}{2m}.$$

Energy difference for these two energy levels is,

$$E(2, 1, 1) - E(1, 1, 1) = 6\frac{\pi^2h^2}{2m} - 3\frac{\pi^2h^2}{2m}$$

$$= 3\frac{\pi^2h^2}{2m}.$$
12. A free particle has a wave function,

\[ \Psi(x, t) = Ae^{i(2.5 \times 10^{11}x - 2.1 \times 10^{13}t)}, \]

where \( x \) is in metres and \( t \) is in seconds. What is the mass of the particle?

(a) Mass can only be determined if \( A \) is known.

(b) 0.012 kg.  
(c) 0.11 kg.  
(d) 5.7 \times 10^{-16} \text{ kg.}  
(e) 1.7 \times 10^{15} \text{ kg.}

**Answer 12:**

As we know that general equation of wave function is,

\[ \Psi(x, t) = Ae^{ikx-\omega t}. \]

Comparison of this equation with the given wave equation of for the free particle yields,

\[ k = 2.5 \times 10^{11} \text{ m}^{-1} \]
\[ \omega = 2.1 \times 10^{13} \text{ s}^{-1}. \]

Mass of the particle can be calculated by the dispersion relation,

\[ \omega^2 = \frac{k}{m} \]
\[ \Rightarrow m = \frac{k}{\omega^2} = \frac{2.5 \times 10^{11}}{(2.1 \times 10^{13})^2} = 5.7 \times 10^{-16} \text{ kg.} \]

Hence option (d) is the correct answer.

You don’t really need to remember the dispersion relationship. Look at the TDSE in the absence of \( V \):

\[ \text{TDSE : } -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \Psi(x, t) = i\hbar \frac{d}{dt} \Psi(x, t). \]

Inserting the supplied wave function into the above, the relationship \( \omega^2 = k/m \) automatically pops out. Students are tempted to use \( E = \hbar \omega \) and \( E = p^2/2m \). The
former relationship does not hold for all particles, it is specific in its meaning—it says that energy of a photon $E$ is related to the frequency of the electromagnetic wave associated with the photon. Blindly using these relations is wrong!

13. A magic-eye was shown as a classroom demonstration. In this gadget, a beam of electrons is emitted from a cathode and moves horizontally towards a bowl shaped anode. When viewed from top, the electrons produced a characteristics glow as shown.

If the tube is placed inside a perpendicular magnetic field pointing out of the plane of the paper, as shown in Fig (b), the view from the top will look like:

**Answer 13:**

Let’s find out the direction of the force $\vec{F}$ on the beam of electrons. The force is given by $\vec{F} = -e\vec{v} \times \vec{B}$. Using the right hand rule; pointing the fingers in the direction of $\vec{v}$, and curling these into the direction of $\vec{B}$, the thumb points against the direction
of force. (Against because of the negative sign, the electrons are negatively charged). In the leftward beam, the beam bends upwards and in the rightward beam, it bends downwards. Hence option (d) is the correct answer.

14. A beam of electrons accelerated through a potential difference \( V_0 \) is directed at a single slit of width \( a \), then detected at a screen at a distance \( L \) beyond the slit. How far from a point directly in the line of the beam is the first location where no electrons are ever detected?

(a) \( L/(a\sqrt{V}) \).  
(b) \( Lh/(a\sqrt{2meV}) \).

(c) The electrons are detectable everywhere.

(d) \( L/a \).  
(e) \( Lh/(2a\sqrt{2meV}) \).

**Answer 14:**

Path difference between ray I and II is,

\[
\text{Path difference} = \frac{a}{2} \sin \theta,
\]

and for destructive interference this path difference must be \( \lambda/2 \).

\[
\frac{a}{2} \sin \theta = \frac{\lambda}{2} \\
\sin \theta = \lambda
\]

As \( \theta \) is small, \( \sin \theta \approx \tan \theta = \frac{z}{L} = \frac{\lambda}{a} \)

\[
z = \frac{L\lambda}{a}.
\]

Since \( \lambda = \frac{h}{\sqrt{2meV}} \), \( z = \frac{L}{a} \frac{h}{\sqrt{2meV}} \) Hence option (b) is the correct answer.
15. An electron of energy 1 eV is trapped inside an infinite well of length 30 cm. What is the distance between two consecutive nodes of the electron’s wavefunction? (A node is a point where the wavefunction goes to zero.)

(a) There are no nodes in the electron’s wavefunction.

(b) The distance between consecutive nodes is zero.

(c) $1.25 \times 10^{-18}$ m.

(d) $6 \times 10^{-10}$ m.

(e) None of the above.

**Answer 15:**

We are given that,

$$\text{Energy of electron } = E = 1 \text{ eV} = 1.6 \times 10^{-19} \text{ J}$$

$$\text{Length of infinite well } = L = 30 \text{ cm} = 0.3 \text{ m}.$$  

The wave function is $A \sin \left( \frac{n\pi x}{L} \right)$. The consecutive nodes differ in space by $\Delta x$. At a node, the argument of the sine function must differ in phase by $\pi$. Hence $\frac{n\pi \Delta x}{L} = \pi$, $\Rightarrow \Delta x = L/n$. We just need to find $n$, the state of the wave function.
The quantum number \( n \) is calculated as,

\[
E_n = \frac{n^2 \hbar^2 \pi^2}{2mL^2}
\]

\[
\Rightarrow n^2 = \frac{2EmL^2}{\hbar^2 \pi^2}
\]

\[
n = \sqrt{\frac{2EmL^2}{\hbar^2 \pi^2}}
\]

\[
= \sqrt{\frac{2 \times 1.6 \times 10^{-19} \text{ J} \times 9.11 \times 10^{-31} \text{ kg} \times (0.3 \text{ m})^2}{(1.06 \times 10^{-34} \text{ Js})^2 \pi^2}}
\]

\[
= \sqrt{2.4 \times 10^{17}}
\]

\[
= 4.9 \times 10^8.
\]

Thus the distance between two consecutive nodes is,

\[
\Delta x = \frac{0.3 \text{ m}}{4.9 \times 10^8}
\]

\[
= 6 \times 10^{-10}.
\]

Hence option (d) is the correct answer.

16. For the electron in the previous question, will quantum effects be visible? Please give a reason. \[2 \text{ Marks}\]

**Answer 16:**

Since the length of infinite well is very large i.e. 30 cm, for a small amount of energy 1 eV, the number of nodes will be very large i.e. \( 4.9 \times 10^8 \). Since the number of nodes is very large, the waves are “squeezed” close together, the de Broglie wavelength is extremely small obscuring chances of observing the quantum wave behavior at the classical macroscopic scale. At such a high value of \( n \), quantum effects are not visible.

Another way of looking at this is that the wave function is such that the probability of finding the electron becomes equal everywhere, i.e. it imparting the electron a continuous quality rather than quantized. All of this ties in well with Bohr’s corresponding principle.

17. An electron is trapped in an infinite well of length \( L \) and ground state energy \( E_1 \). At \( t = 0 \), the wavefunction is,

\[
\Psi(x, 0) = \frac{1}{\sqrt{5L}} (\psi_1(x) + 2\psi_2(x)),
\]
where \( \psi_1(x) \) and \( \psi_2(x) \) are normalized wavefunctions in the ground and first excited states. The wavefunction at \( t = \frac{\pi \hbar}{E_1} \) is given by:

\[
\Psi(x, t) = \frac{1}{\sqrt{5L}} (\psi_1(x) + 2\psi_2(x) e^{-iE_1t/\hbar})
\]

**Answer 17:**

We are given that,

At \( t = 0 \) \[
\Psi(x, 0) = \frac{1}{\sqrt{5L}} (\psi_1(x) + 2\psi_2(x))
\]

At any time \( t \) \[
\Psi(x, t) = \frac{1}{\sqrt{5L}} (\psi_1(x)e^{-iE_1t/\hbar} + 2\psi_2(x)e^{-iE_2t/\hbar}).
\]
If $E_1$ is the energy of ground state, energy of first excited state will be $E_2 = 4E_1$.

$$\Rightarrow \Psi(x, t) = \frac{1}{\sqrt{5L}}(\psi_1(x)e^{-iE_1t/\hbar} + 2\psi_2(x)e^{-i4E_1t/\hbar})$$

At $t = \frac{\pi \hbar}{E_1}$

$$\Psi(x, t) = \frac{1}{\sqrt{5L}}(\psi_1(x)e^{-i\pi} + 2\psi_2(x)e^{-i4\pi})$$

$$= \frac{1}{\sqrt{5L}}(\psi_1(x)(-1) + 2\psi_2(x)(+1))$$

$$= \frac{1}{\sqrt{5L}}(-\psi_1(x) + 2\psi_2(x)).$$

Now we need to find out what $-\psi_1(x) + 2\psi_2(x)$ looks like. The construction is seen in the series of diagrams below.

Hence option (a) is the correct answer.

18. The potential energy profile in a certain region is shown.
A particle of energy $E$ exists inside this region. A sketch of the possible (real part) of the wavefunction is;

(e) None of the above.

**Answer 18:**

Option (d) is the correct answer. The wavefunction is zero at $x \geq b$ because of the infinite potential and extends into the region $x \leq a$. Furthermore, the value of $k$ increases and wavelength decreases as we go from $x = a$ to $x = b$.

19. Suppose a particle is in the ground state with wavefunction $\psi_1(x)$. Which one of the following is the probability that the particle will be found in a narrow range between $x$ and $x + dx$. Furthermore, the value of $k$ increases and wavelength decreases as we go from $x = a$ to $x = b$.

(a) $|\psi_1(x)|^2 dx$.

(b) $x|\psi_1(x)|^2 dx$. 
(c) \( \int x|\psi_1(x)|^2dx \).

(d) \( \int_{-\infty}^{+\infty} x|\psi_1(x)|^2dx \).

(e) None of the above.

**Answer 19:**

Option (a) is the correct answer.

20. At time \( t = 0 \), the state for a particle inside an infinite well is \( \frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \), where \( \psi_1 \) and \( \psi_2 \) are ground and first excited states: with energies \( E_1 \) and \( E_2 \) respectively.

We first measure the position of the particle at time \( t = 0 \) and obtain the result \( x_0 \).

Immediately after the position measurement, we measure the energy. What possible result(s) can we obtain for the energy measurement?

(a) We can only measure either \( E_1 \) or \( E_2 \).

(b) We can obtain one of the energy values \( E_n = \frac{n^2 \hbar^2}{2ma^2} \), where \( n \) can be an arbitrary large integer.

(c) We can only measure \( \frac{1}{2}(E_1 + E_2) \).

(d) We may measure any energy \( E = \sum_{n=1}^{\infty} C_n E_n \), where \( C_n \) are coefficients so that \( \sum_{n=1}^{\infty} |C_n|^2 = 1 \).

(e) None of the above.

**Answer 20:**

The correct option is (a).

21. A free particle has a wavefunction \( A(e^{ikx} + e^{-ikx}) \) and energy \( E \). \( A \) is a normalization constant. Mark **True** of **False** against these statements.

(a) The probability density does not change with time.

(b) The probability density is constant in space \( x \).

(c) The de Broglie wave associated with the particle is in fact a standing wave.

[6 Marks]
**Answer 21:**

We are given the wavefunction of free particle,

\[ \Psi(x, t) = A(e^{ikx} + e^{-ikx})e^{-i\frac{Et}{\hbar}} \]

\[ \Rightarrow \Psi^*(x, t) = A^\ast(e^{-ikx} + e^{ikx})e^{\frac{Et}{\hbar}} \]

\[ \Psi^*(x, t)\Psi(x, t) = A^\ast(e^{-ikx} + e^{ikx}) \cdot A(e^{ikx} + e^{-ikx})e^{-i\frac{Et}{\hbar}}e^{i\frac{Et}{\hbar}} \]

\[ = A^2(1 + e^{-2ikx} + e^{2ikx} + 1) \]

\[ = A^2(2 + 2\cos(2kx)) \]

Using \( \cos x = \frac{e^{ix} + e^{-ix}}{2} \)

\[ |\Psi^2(x, t)|^2 = A^2(2 + 2\cos(2kx)) \]

\[ = A^2(2 + 2\cos(2kx)) \]

\[ = 2A^2(1 + \cos(2kx)) \]

\[ = 2A^2 \cdot 2\cos^2(kx) \]

\[ = 4A^2 \cdot \cos^2(kx) \]

(a) **True** since \( p(x) = |\Psi^2(x, t)|^2 \) is independent of time.

(b) **False** since \( p(x) \) depends on \( x \) and changes with \( x \).

(c) **True** because the forward and backward propagating waves have equal amplitudes and the probability density does not change with time.

22. The uncertainty relationship for a particle moving in a straight line is \( \Delta p \Delta x \geq \hbar/2 \).
If the particle is moving in a circle with angular momentum \( L \), the uncertainty relationship becomes:

(HINT: Distance becomes the arc length!)

(a) \( \Delta L \Delta \theta \geq \frac{\hbar}{2} \).
(b) \( \Delta L \Delta S \geq \frac{\hbar}{2} \).
(c) \( \Delta L \Delta R \geq \frac{\hbar}{2} \).
(d) \( \Delta L \Delta \theta \leq \frac{\hbar}{2} \).
(e) None of the above.

Answer 22:

According to uncertainty principle,

\[ \Delta p \Delta x \geq \frac{\hbar}{2}. \]

If particle moves in a circle of radius \( r \) and angular momentum \( L \), then

\[ L = pr \]
\[ \Rightarrow \Delta L = \Delta pr \]
\[ \Rightarrow \Delta p = \frac{\Delta L}{r} \]
and \( \Delta x = r \Delta \theta \).

Using these values of \( \Delta p \) and \( \Delta x \) in uncertainty relation,

\[ \frac{\Delta L}{r} \cdot r \Delta \theta \geq \frac{\hbar}{2} \]
\[ \Delta L \Delta \theta \geq \frac{\hbar}{2}. \]

Hence option (a) is the correct answer.

23. In a scanning tunneling microscope (STM), the tunneling probability of electrons from metal surface to a probe tip is proportional to \( \exp(-2\alpha L) \), where \( L \) is the tip-sample distance and \( \alpha = 1 \text{ nm}^{-1} \) is the inverse of the penetration length.
If the tip moves closer to the surface by $\Delta L = 0.1 \text{ nm}$, the tunneling current,

(a) remains unchanged.

(b) increase by 22 %.

(c) decrease by 22 %.

(d) increase by 10 %.

(e) decrease by 10 %.

**Answer 23:**

We are given that,

$$\text{Tip-sample distance} = \alpha = 1 \text{ nm}^{-1} = 1 \times 10^9 \text{ m}^{-1}$$

$$\text{Distance covered by tip} = \Delta L = 0.1 \text{ nm} = 0.1 \times 10^{-9} \text{ m}.$$  

Tunneling probability is,

$$T_i = e^{-2\alpha L}.$$  

If tip moves closer to the surface by $\Delta L$, final tunneling probability will become,

$$T_f = e^{-2\alpha (L-\Delta L)}.$$  

ratio of tunneling probabilities is,

$$\frac{T_f}{T_i} = \frac{e^{-2\alpha (L-\Delta L)}}{e^{-2\alpha L}} = e^{2\alpha \Delta L} = e^{2 \times 10^9 \times 0.1 \times 10^{-9}} = e^{0.2} = 1.22.$$  

Hence there is a 22\% increase in the tunneling current and the correct answer is (b).
Final Exam 2013: Modern Physics

PART B

Attempt all questions.

1. The Heisenberg uncertainty principle applies to photons as well as to material particles. Thus a photon confined to a small box of size $\Delta x$ necessarily has a large uncertainty in momentum and uncertainty in energy. Recall that for a photon $E = pc$.

(a) Estimate the uncertainty in energy for a photon confined to the tiny box of size $\Delta x$. [2 Marks]

Answer 1 (a):

We are given that,

$$E = pc.$$  

Uncertainty in energy is,

$$\Delta E \approx c \Delta p.$$  \hspace{1cm} (1)

Uncertainty in momentum $\Delta p$ for a photon confined to the tiny box of size $\Delta x$ can be calculated by using the uncertainty relation,

$$\Delta x ~ \Delta p \geq \frac{\hbar}{2},$$

where $\hbar$ is the reduced Planck’s constant.

$$\Rightarrow \Delta p \geq \frac{\hbar}{2\Delta x}.$$
Using this value of $\Delta p$ in equation (1) yields,

$$\Delta E \sim c \left( \frac{\hbar}{2\Delta x} \right)$$

$$\Delta E \sim \frac{c\hbar}{2\Delta x}$$

(b) If $\Delta E \sim E$, what is the effective mass of the photon? [2 Marks]

**Answer 1 (b):**

We are given that,

$$\Delta E \sim E$$

$$\Rightarrow E = \frac{c\hbar}{2\Delta x}$$

Effective mass can be calculated by using energy-mass relationship,

$$E = m_{\text{eff}}c^2,$$

where $m_{\text{eff}}$ is the effective mass of the photon.

$$\Rightarrow m_{\text{eff}} = \frac{E}{c^2} = \frac{c\hbar}{2\Delta x} \cdot \frac{1}{c^2} = \frac{\hbar}{2c\Delta x}.$$

(c) This mass can be extremely large, if $\Delta x$ is tiny. If $\Delta x$ is sufficiently small, the large mass can create a large gravitational field, sufficiently large to form a black hole. When this happens $\Delta x$ is called the Planck length, and this is when gravity and quantum mechanics become intermixed. For a black hole, not even light can escape.
Consider a star of mass $M$ and radius $R$ as shown above. If an object is to be launched from the star’s surface so that it escapes the star’s gravitational pull, it needs a minimum velocity $v_{\text{esc}}$ called the escape velocity. Show that $v_{\text{esc}} = \sqrt{\frac{2GM}{R}}$. [4 Marks]

**Answer 1 (c):**

We are given that,

- Mass of the star $= M$
- Radius of the star $= R$
- Let mass of the object $= m$,

and let speed of the object to escape from the star’s gravitational pull is $v_{\text{esc}}$. In order to escape from the gravitational pull of star, kinetic energy of the object should be at least equal to the gravitational potential energy.

\[
\frac{1}{2}mv_{\text{esc}}^2 = G\frac{mM}{R}
\]

\[
v_{\text{esc}}^2 = \frac{2GM}{R}
\]

\[
v_{\text{esc}} = \sqrt{\frac{2GM}{R}}.
\]
Which is the required result.

(d) If \( v_{esc} = c \), nothing can escape from this star, not even light. If we were to replace the star of mass \( M \) with a photon of the mass calculated in part (b), and confined to length \( \Delta x \), and set \( R = \Delta x \), calculate the Planck length in terms of \( G, \hbar \) and \( c \). [3 Marks]

**Answer 1 (d):**

Using the relation of \( v_{esc} \) derived in part (c)

\[
v_{esc} = \sqrt{\frac{2GM}{R}},
\]

where \( v_{esc} = c \),

\[
c = \sqrt{\frac{2GM}{R}}
\]

\[
c^2 = \frac{2GM}{R}.
\]

Setting \( R = \Delta x \) and \( M = m_{eff} \),

\[
c^2 = \frac{2Gm_{eff}}{\Delta x} = \frac{2G}{\Delta x} \cdot \frac{\hbar}{2c\Delta x} = \frac{G\hbar}{c}\Delta x^2
\]

\[
\Delta x^2 = \frac{G\hbar}{c^3}
\]

\[
\Delta x = \sqrt{\frac{G\hbar}{c^3}}.
\]

(e) If \( G = 6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2 \), find the numerical value of Planck’s length. [2 Marks]
Answer 1 (e):

\[ \Delta x = \sqrt{\frac{G\hbar}{c^3}} \]
\[ = \sqrt{\frac{6.67 \times 10^{-11} \text{ N m}^2/\text{kg}^2 \times 1.06 \times 10^{-34} \text{ Js}}{(3.0 \times 10^8 \text{ ms}^{-1})^3}} \]
\[ = \sqrt{2.62 \times 10^{-70} \text{ m}^2} \]
\[ = 1.6 \times 10^{-35} \text{ m}. \]

(f) What is the diameter of a proton (about 2 fm = 2 \times 10^{-15} \text{ m}) in units of Planck’s length? [2 Marks]

Answer 1 (f):

We are given that,

Diameter of photon = 2 fm

\[ \left( \frac{\text{Diameter of photon}}{\text{in Planck’s length}} \right) = \frac{2 \times 10^{-15} \text{ m}}{1.6 \times 10^{-35} \text{ m/Planck length}} \]
\[ = 1.24 \times 10^{20} \text{ Planck lengths}. \]

2. The radioactive decay of certain heavy nuclei by emission of an alpha particle is a result of quantum tunneling. Imagine an alpha particle moving around inside a nucleus, such as thorium (mass number= 232).

When the alpha particles bounces against the surface of the nucleus, it meets a barrier caused by the attractive nuclear force. The dimensions of barrier vary a lot from one nucleus to another, but as representative numbers you can assume that the barrier’s width is \( L \approx 35 \text{ fm} \) (1 fm = 10^{-15} \text{ m}) and the average barrier height is such that \( V_0 - E \approx 5 \)
MeV. Find the probability that an alpha hitting the nucleus surface will escape. Given that the alpha hits the nuclear surface about $5 \times 10^{21}$ times per second, what is the probability that it will escape in a day?

The tunneling probability is $T = e^{-2\alpha L}$ where $\alpha = \sqrt{2m(V_0 - E)/\hbar}$ and $L$ is the barrier length. (1 MeV $= 10^6$ eV). [5 Marks]

**Answer 2:**

We are given that,

Barrier length $= L = 35$ fm $= 35 \times 10^{-15}$ m

Barrier height $= (V_0 - E) = 5$ MeV $= 5 \times 10^6 \times 1.6 \times 10^{-19}$ J

$= 8 \times 10^{-13}$ J

Tunneling probability $= T = e^{-2\alpha L}$,

where $\alpha = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$

and mass of the nucleus $= m = 4 \times 1.67 \times 10^{-27}$ kg $= 6.68 \times 10^{-27}$ kg.

In order to calculate probability of escape $T$, let’s first calculate $\alpha$.

$$\alpha = \frac{\sqrt{2m(V_0 - E)}}{\hbar}$$

$$= \frac{\sqrt{2 \times 6.68 \times 10^{-27} \text{ kg} \times 5 \times 10^6 \times 1.6 \times 10^{-19} \text{ J}}}{1.06 \times 10^{-34} \text{ Js}}$$

$$= 9.75 \times 10^{14} \text{ m}^{-1}.$$ 

Tunneling probability is,

$$T = e^{-2\alpha L}$$

$$= e^{-2 \times 9.75 \times 10^{14} \text{ m}^{-1} \times 35 \times 10^{-15} \text{ m}}$$

$$= e^{-68.25}$$

$$= 2.29 \times 10^{-30}.$$
The probability that an alpha particle hitting the nucleus surface will escape is $2.29 \times 10^{-30}$.

If no. of hits per second $= 5 \times 10^{21}/s$

\[
\text{Probability of escape in a day} = T = 2.29 \times 10^{-30} \times 5 \times 10^{21}/s \times 24 \times 60 \times 60 \ s
\]

$= 9.89 \times 10^{-4}$

$= 9.89 \times 10^{-2} \%$