Solution: Midterm Exam

1. An experiment to measure the Planck's constant h gives it in the form $h = K\lambda^{1/3}$, where K is a constant known exactly and λ is the measured wavelength emitted by a hydrogen lamp. If a student has measured λ with a fractional uncertainty of 0.3% and his best estimate for h is 6.644×10^{-34} Js. The final answer for h is, (3points)

   (a) $h = (6.644 \pm 0.071) \times 10^{-34}$ Js, (1%).

   (b) $h = (6.644 \pm 0.008) \times 10^{-34}$ Js, (3%).

   (c) $h = (6.644 \pm 0.081) \times 10^{-34}$ Js, (1.2%)

   (d) $h = (6.644 \pm 0.007) \times 10^{-34}$ Js, (0.1%).

   (e) $h = (6.644 \pm 0.092) \times 10^{-34}$ Js, (1%).

   Solution:

   The correct answer is (**d**).

   Given expression for Planck's constant is,

$$h = K\lambda^{1/3}. \tag{1}$$

   To find uncertainty in the above expression, we need to apply Taylor series approximation,

$$\Delta h = \sqrt{\left(\frac{dh}{d\lambda}\Delta\lambda\right)^2},$$

   implying,

$$\Delta h = \left(\frac{1}{3}\right)K\lambda^{\frac{1}{3}-1}\Delta\lambda,$$

   and can be written as,

$$\Delta h = \frac{K\lambda^{1/3}\Delta\lambda}{3\lambda} = K\lambda^{1/3}\left(\frac{1}{3}\frac{\Delta\lambda}{\lambda}\right),$$

   Substituting Equation (1) in the above expression gives,

$$\Delta h = h\left(\frac{1}{3}\frac{\Delta\lambda}{\lambda}\right).$$

Now if we substitute the given values of measurands, we are left with,

$$\begin{aligned}\Delta h &= h\left(\frac{1}{3}\frac{0.3}{100}\right) = h \times (1 \times 10^{-3}),\\ &= (6.644 \times 10^{-34}) \times (1 \times 10^{-3}) = 6.644 \times 10^{-37}\,\text{Js},\\ &= 0.007 \times 10^{-34}\,\text{Js}.\end{aligned}$$

and the fractional uncertainty is,

$$\frac{\Delta h}{h} = 0.001 = 0.1\%.$$

Hence, we can quote our final result as,

$$h = (6.644 \pm 0.007) \times 10^{-34}\,\text{Js}.$$

2. With a good stop watch and some practice, you can measure times and uncertainty ranging from approximately from 1 s to many minutes with an uncertainty of 0.1 s or so. Suppose we wish to find the time period τ of a pendulum with $\tau \approx 0.5\,\text{s}$. If we time 1 oscillation, we have an uncertainty of approximately 20%. If we measure the total time for 5 oscillations and get $(2.4 \pm 0.1\,\text{s})$, the final answer for τ along with its absolute and percentage uncertainties is. (3 points)

(a) $\tau = (0.48 \pm 0.10)\,s, (21\%)$.

(b) $\tau = (0.52 \pm 0.02)\,s, (4\%)$.

(c) $\tau = (0.48 \pm 0.02)\,s, (4\%)$.

(d) $\tau = (0.52 \pm 0.01)\,s, (2\%)$.

(e) $\tau = (0.48 \pm 0.02)\,s, (10\%)$.

Solution:

The correct answer is (**c**).

The time period of a pendulum $\tau \approx 0.5\,\text{s}$. The uncertainty is given as,

$$\frac{\Delta \tau}{\tau} = 20\% = \frac{20}{100} = 0.2\,\text{s}.$$

The total time for 5 oscillations is given as,

$$t = (2.4 \pm 0.1)\,s$$

The time period of the pendulum can be found out as,

$$\tau = \frac{t}{5} = \frac{2.4}{5} = 0.48\,\text{s}.$$

The uncertainty can be found out using the Taylor series approximation,

$$\Delta\tau = \sqrt{\left(\frac{\partial \tau}{\partial t}\Delta t\right)^2} = \left(\frac{1}{5}\right)\Delta t,$$
$$= \left(\frac{1}{5}\right)(0.1) = 0.02\,\text{s}.$$

The fractional uncertainty in the time period is,

$$\frac{\Delta\tau}{\tau} = \frac{0.02}{0.48} = 0.042,$$
$$= 4\%$$

The final value can be written as,

$$\tau = (0.48 \pm 0.02)\,\text{s}.$$

3. A zener diode with $V_z = 6\,\text{V}$ operated at a current of $I_z = 1\,\text{mA}$ has a dynamic resistance ($dV_z/dI_z = 3\,\Omega$). If the current varies by 2%, the fractional uncertainty in the voltage is, (3 points)

(a) $\Delta V_z/V_z = 6 \times 10^{-6}$.

(b) $\Delta V_z/V_z = 1 \times 10^{-5}$.

(c) $\Delta V_z/V_z = 1 \times 10^{-6}$.

(d) $\Delta V_z/V_z = 6 \times 10^{-5}$.

(e) None of the above.

Solution:

The correct answer is (**b**).

The fractional uncertainty in the current value is given as,

$$\frac{\Delta I_z}{I_z} = 2\% = \frac{2}{100} = 0.02,$$

and we can write I_z as,

$$I_z = (1.00 \pm 0.02)\,\text{mA}.$$

We can find uncertainty in voltage using Taylor series approximation,

$$\Delta V_z = \sqrt{\left(\frac{dV_z}{dI_z}\Delta I_z\right)^2},$$
$$= (3) \times (0.02 \times 10^{-3}) = 6 \times 10^{-5}\,\text{V},$$

and the fractional uncertainty is,

$$\frac{\Delta V_z}{V_z} = \frac{6 \times 10^{-5}}{6} = 1 \times 10^{-5}.$$

4. A student has to measure several unknown charges Q on a capacitor by discharging it through a ballistic galvanometer. The discharge kicks the galvanometer's needle, whose resulting maximum swing tells the student the value of Q. She arranges to charge the capacitor to exactly the same voltage (and hence the same charge) six times and to measure the and to measure the resulting charge. Her results in (microcoulombs μC) are,

   1.2 1.4 1.6 1.6 1.2 1.5

   Based on these results, quote the best estimated value of Q alongwith its uncertainty within 68% coverage probability. (3 points)

   (a) $(1.42 \pm 0.14)\,\mu$C.

   (b) $(2.2 \pm 0.4)\,\mu$C.

   (c) $(1.42 \pm 0.07)\,\mu$C.

   (d) $(1.6 \pm 0.2)\,\mu$C.

   (e) None of the above.

   <u>Solution:</u>

   The correct answer is (**c**).

   The best estimated value of the charge Q stored on a capacitor is,

$$< Q > = \frac{\sum_{i=1}^{6} Q_i}{n} = \frac{8.50}{6} = 1.417\,\mu\text{C}.$$

Deviations can be calculated using the following expression,

$$d_i = Q_i - < Q >,$$

and tabulated in Table (I).

Charge, $Q\,(\mu\,\mathrm{C})$	Deviations, $d_i\,(\mu\,\mathrm{C})$	Deviations squared, $d_i^2\,(\mu\,\mathrm{C})^2$
1.2	−0.2167	0.0469
1.4	−0.0167	0.0003
1.6	0.1833	0.0336
1.6	0.1833	0.0336
1.2	−0.2167	0.0469
1.5	0.0833	0.0069

TABLE I: Table for calculated deviations.

The standard deviation can be calculated as,

$$s = \sqrt{\frac{\sum_{i=1}^{6} d_i^2}{n}} = \sqrt{\frac{0.1683}{6}} = 0.167\,\mu\mathrm{C}.$$

The standard uncertainty can be found out as,

$$\sigma = \sqrt{\frac{n}{n-1}}\,(s) = \sqrt{\frac{6}{5}}\,(0.167) = 0.183\,\mu\mathrm{C},$$

and the standard uncertainty in the mean value is,

$$\sigma_m = \frac{\sigma}{\sqrt{n}} = \frac{0.183}{\sqrt{6}} = 0.074\,\mu\mathrm{C}.$$

The best estimated value of the charge Q can be quoted as,

$$Q = (1.42 \pm 0.07)\,\mu\mathrm{C}.$$

We are considering only one standard uncertainty (σ_m) which corresponds to 68% coverage probability. Hence, we conclude that our result lies within 68% confidence of interval.

5. Three groups of particle physicists measure the mass of a certain elementary particle with results,

$$\begin{aligned} \text{Physicist 1 :} \quad m &= (1967.0 \pm 0.1)\,\text{Mev/c}^2. \\ \text{Physicist 2 :} \quad m &= (1969.0 \pm 1.4)\,\text{Mev/c}^2. \\ \text{Physicist 3 :} \quad m &= (1972.1 \pm 2.5)\,\text{Mev/c}^2. \end{aligned}$$

If they decide to combine their results, the respective weights and the best estimated of m through weighted average would be, (3 points)

(a) Weights $= 100, 0.51, 0.16$, Weighted average $= 1967\,\text{Mev/c}^2$.

(b) Weights $= 100, 0.5, 0.2$, Weighted average $= 1971.4\,\text{Mev/c}^2$.

(c) Weights $= 100, 0.50, 0.16$, Weighted average $= 1981.4\,\text{Mev/c}^2$.

(d) Weights $= 100, 0.4, 0.2$, Weighted average $= 1965.3\,\text{Mev/c}^2$.

(e) None of the above.

Solution:

The correct answer is (**a**).

The weight w_i of each measurement is the reciprocal square of the corresponding uncertainty σ_i, that is,

$$w_i = \frac{1}{\sigma_i^2}.$$

For the given measurands, the weights can be found out as,

$$\begin{aligned} \text{Physicist 1 :} \quad w_1 &= \left(\frac{1}{(0.1)^2}\right) = 100, \\ \text{Physicist 2 :} \quad w_2 &= \left(\frac{1}{(1.4)^2}\right) = 0.51, \\ \text{Physicist 3 :} \quad w_3 &= \left(\frac{1}{(2.5)^2}\right) = 0.16. \end{aligned}$$

The best estimated value of the mass of an elementary particle can be found out by the method of weighted average and given as,

$$\begin{aligned} m_{avg} &= \frac{\sum_{i=1}^{3} w_i m_i}{\sum_{i=1}^{3} w_i}, \\ &= \frac{(100 \times 1967) + (0.51 \times 1969) + (0.16 \times 1972.1)}{100 + 0.51 + 0.16} = 1967\,\text{Mev/c}^2. \end{aligned}$$

6. Suppose we measure the mass of a ball six times using a digital weighing balance (rating = 0) and get the following set of values (in grams g),

$$24.25,\ 24.26,\ 24.22,\ 24.24,\ 24.25,\ 24.26$$

The standard uncertainty in the mean value is $\sigma_m = 0.006\,\text{g}$. How would you quote your final result? (3 points)

Solution:

The mass is measured using a digital weighing balance and by looking at the data, we can tell that the balance has a resolution of 0.01 g. Now we'll associate a uniform probability distribution function with it and the scale uncertainty is given as,

$$\sigma_{\text{scale}} = \frac{0.005}{\sqrt{3}} = 0.003\,\text{g}.$$

This is the type B uncertainty associated with a single measurement.
Since the measurement is repeated 6 times, it has some statistics involved and type A uncertainty associated with it. The mean value of the measurand can be found out,

$$\begin{aligned} < m > &= \frac{\sum_{i=1}^{6} m_i}{n}, \\ &= \frac{24.25 + 24.26 + 24.22 + 24.24 + 24.25 + 24.26}{6}, \\ &= 24.247\,\text{g}, \end{aligned}$$

and the standard uncertainty in the mean value is given $\sigma_m = 0.006\,\text{g}$.
Now we will add type A and type B uncertainties in the same quadrature,

$$\begin{aligned} \sigma_{\text{total}} &= \sqrt{\sigma_{\text{scale}}^2 + \sigma_m^2}, \\ &= \sqrt{(0.003)^2 + (0.006)^2} = 0.0067\,\text{g}. \end{aligned}$$

The value of the mass can be quoted as,

$$m = (24.247 \pm 0.007)\,\text{g}.$$

7. If a stone is thrown vertically upward with speed v, it should rise to a height h given by $v^2 = 2gh$. A student measures v and h for several throws and come up with results tabulated in Table (1).

Height h (m)	0.4	0.8	1.4	2.0	2.6	3.4	3.8
Speed v (m/s)	2.64	4.12	5.00	6.16	6.71	7.87	8.48

TABLE II: Experimental data for height h and speed v.

(a) Use mathematical expressions of least squares fitting of a straight line with equal weights to find the best estimated value of g. (7 points)

(b) Calculate uncertainty in g and quote your final result. Assume h and v are measured with zero uncertainties. (5 points)

Solution:

(a) The relationship between height and speed is,

$$v^2 = 2gh,$$

The v^2 term has a linear dependence on height h, hence we can take v^2 as our dependent variable and height h as an independent variable. The slope of the above equation is equal to $2g$.

The mean values of the measurands are,

$$\begin{aligned}\bar{v^2} &= \frac{\sum_{i=1}^{7} v_i^2}{n} \\ &= \frac{6.97 + 16.97 + 25.00 + 37.94 + 45.02 + 61.94 + 71.91}{7} \\ &= 37.96\,\mathrm{m^2/s^2}.\end{aligned}$$

$$\begin{aligned}\bar{h} &= \frac{\sum_{i=1}^{7} h_i}{n} \\ &= \frac{0.4 + 0.8 + 1.4 + 2.0 + 2.6 + 3.4 + 3.8}{7} \\ &= 2.06\,\mathrm{m}.\end{aligned}$$

The best estimated value of the slope can be found out using the relationship,

$$m = \frac{\sum_{i=1}^{N} y_i(x_i - \bar{x})}{\sum_{i=1}^{N} (x_i - \bar{x})^2}.$$

Utilizing our dependent and independent variables, the slope can be found out as,

$$m = \frac{\sum_{i=1}^{7} v_i^2(h_i - \bar{h})}{\sum_{i=1}^{7}(h_i - \bar{h})^2},$$

Implying,

$$\begin{aligned} m &= \frac{[6.97(0.4-2.06) + 16.97(0.8-2.06) + 25.00(1.4-2.06)}{[(0.4-2.06)^2 + (0.8-2.06)^2 + (1.4-2.06)^2 + (2.0-2.06)^2}, \\ &= \frac{+37.94(2.0-2.06) + 45.02(2.6-2.06) + 61.94(3.4-2.06) + 71.91(3.8-2.06)]}{+(2.6-2.06)^2 + (3.4-2.06)^2 + (3.8-2.06)^2]}, \\ &= \frac{181.46}{9.90}, \\ &= 18.33\,\text{m/s}^2. \end{aligned}$$

Since slope is equal to $2g$, we can write,

$$\text{Slope} = 2g = 18.33\,\text{m/s}^2,$$

and the value of g can be deduced as,

$$g = \frac{18.33}{2} = 9.165\text{m/s}^2.$$

(b) The intercept can be calculated as,

$$c = \bar{y} - m\bar{x}.$$

Hence,

$$\begin{aligned} c &= \bar{v^2} - m\bar{h} = 37.96 - (18.33)(2.06), \\ &= 0.20\,\text{m}^2/\text{s}^2. \end{aligned}$$

Uncertainty in slope m is given as,

$$u_m = \sqrt{\frac{\sum_i^N d_i^2}{D(N-2)}},$$

where,

$$\begin{aligned} d_i &= y_i - mx_i - c, \\ D &= \sum_i^N (x_i - \bar{x})^2. \end{aligned}$$

Now,

$$
\begin{aligned}
u_m &= \sqrt{\frac{(-0.56)^2 + (2.11)^2 + (-0.87)^2 + (1.08)^2 + (-2.84) + (-0.60)^2 + (2.04)^2}{5 \times [(-1.66)^2 + (-1.26)^2 + (-0.66)^2 + (-0.06)^2 + (0.54)^2 + (1.34)^2 + (1.74)^2]}}, \\
&= 0.6243\,\mathrm{m/s^2}.
\end{aligned}
$$

The uncertainty in the value of g is,

$$
\begin{aligned}
\Delta g &= \sqrt{\left(\frac{dg}{dm}\Delta m\right)^2}, \\
&= \frac{1}{2}(\Delta m) = \frac{1}{2}(0.62), \\
&= 0.31\,\mathrm{m/s^2}.
\end{aligned}
$$

The final of g can be quoted as,

$$
g = (9.2 \pm 0.3)\,\mathrm{m/s^2}.
$$