Third Collaborative homework

Solution

Note: Deadline is 18 April 2013, 10 am. Place your homework in the orange box on physics floor. Mention names of all collaborators in the group. Work alone or in groups upto five. Answer the questions on the same sheet and return.

1. Fig.(a) shows a 50:50 beamsplitter (BS) with two input ports and two output ports. The input rays have arrows coming into the BS and output rays have arrows going away.

The input photon has two possible paths, labeled by $\psi_1$ and $\psi_2$ (as shown in the fig.(a)). The output photon also has two possible paths, also labeled as $\psi_1$ and $\psi_2$ as shown. The BS creates a superposition i.e., an input field goes into an output field according to the rule given below.

\[
\psi_1 \xrightarrow{\text{BS}} \frac{1}{\sqrt{2}} (\psi_1 + \psi_2), \quad \text{and} \\
\psi_2 \xrightarrow{\text{BS}} \frac{1}{\sqrt{2}} (\psi_1 - \psi_2)
\]

Now answer these questions.
A single photon comes in as $\psi_1$ as shown in Fig.(b). Which of the detectors $D_1$ and $D_2$ clicks?

**Answer (i):**

Since we have a 50:50 beamsplitter, there is an equal chance of any one of the detectors clicking, but only one at a time. This can also be observed by noting that the wavefunction in the region $A$ is $\frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$. Hence the probability of $D_1$ clicking is \( \left( \frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2} \) and of $D_2$ clicking is also \( \left( \frac{1}{\sqrt{2}} \right)^2 = \frac{1}{2} \).

**(ii)** Many photons come in, all in the state $\psi_1$. How will the detectors click? Both at once, both but one after the other, simultaneously...?

**Answer (ii):**

As many photons come in, both the detectors will click with equal probabilities but never simultaneously. Detection collapses the wavefunction which then appears as a
“particle”. The particle—called the photon—cannot be split. So never two detectors will click at the same time.

(iii) Answer questions (i) and (ii) when the photons come in as a field \( \psi_2 \) instead of \( \psi_1 \) as shown in Fig.(c).

Answer (iii):

The answer to (iii) is identical to (i) and (ii). In the region A, the wavefunction is \( \frac{1}{\sqrt{2}}(\psi_1 - \psi_2) \). The minus sign before \( \psi_2 \) does not make any physical difference here because the probability “washes” away the minus sign, i.e. \( (-\frac{1}{\sqrt{2}})^2 = (\frac{1}{\sqrt{2}})^2 = \frac{1}{2} \).
(iv) If a quantum object is in superposition field \((a\psi_1 + b\psi_2)\), the probability that upon collapse, \(\psi_1\) is observed is \(|a|^2 = a^*a\) and the probability that \(\psi_2\) is observed is \(|b|^2 = b^*b\). Clearly \(|a|^2 + |b|^2 = 1\). (Remember that \(a^*\) represents the complex conjugate of \(a\)). Refer again to Fig.(b), but now suppose that our beamsplitter is not an ideal 50:50 device, rather a 70:30 device meaning that \(D_1\) clicks 70% of the times and \(D_2\) clicks 30% of the times. What is then the photon’s superposition state after the beamsplitter?

**Answer (iv):**

The superposition state is such that the coefficients of \(\psi_1\) and \(\psi_2\), when modulus squared, yield 0.7 and 0.3. Hence one possible superposition field for a 70:30 beamsplitter is \(\sqrt{0.7}\psi_1 + \sqrt{0.3}\psi_2\). Other possibilities are,

\[-\sqrt{0.7}\psi_1 - \sqrt{0.3}\psi_2\]
\[i\sqrt{0.7}\psi_1 + i\sqrt{0.3}\psi_2, \text{ and}\]
\[-i\sqrt{0.7}\psi_1 - i\sqrt{0.3}\psi_2.\]

You will immediately notice that in each case, the probabilities come out as correct, e.g., \((\sqrt{0.7})^2 = | - \sqrt{0.7}|^2 = |i\sqrt{0.7}|^2 = | - i\sqrt{0.7}|^2 = 0.7\)

(v)
Let’s turn back to 50:50 beamsplitters. Consider Fig.(d). $B_1$ and $B_2$ are identical beamsplitters acting in the fashion presented in equation (1), while $M_1$ and $M_2$ are perfect mirrors. Show that $D_3$ will click 100% of the time. The labeling is as shown. How will you explain this strange outcome?

**Answer (v):**

When the incident photon emerges from $B_1$, The wavefunction in the region $C$ is,

$$\psi_1 \xrightarrow{B_1} \frac{1}{\sqrt{2}}(\psi_1 + \psi_2).$$

Now the wavefunction in the region $E$ (before collapse),

$$\frac{1}{\sqrt{2}}(\psi_1 + \psi_2) \xrightarrow{B_2} \frac{1}{\sqrt{2}} \left[ \frac{\psi_1 + \psi_2}{\sqrt{2}} + \frac{\psi_1 - \psi_2}{\sqrt{2}} \right]$$

$$= \frac{1}{2}(\psi_1 + \psi_2 + \psi_1 - \psi_2)$$

$$= \frac{1}{2}(2\psi_1)$$

$$= \psi_1.$$

Since the emergent wavefunction is $\psi_1$ close to the region of detectors, only $D_3$ will click with 100% probability. This strange outcome is a result of destructive interference between the component fields resulting in complete annihilation of the field $\psi_2$ in the
region close to the detectors. This interference is caused by the second beamsplitter while the first beamsplitter creates a superposition.

(vi) Now turn to Fig.(e).

A nonlinear medium is placed in one of the “arms” of the apparatus. Our apparatus can be called an interferometer. The nonlinear medium has the property that it takes any input field $\psi$ and adds a phase $\delta$ to it, i.e.,

$$\psi$$ phase medium \rightarrow $$\psi e^{i\delta}$$.

What is the probability that $D_3$ clicks and the probability that $D_4$ clicks? Can we adjust $\delta$ and make $D_3$ and $D_4$ click with equal probabilities?

**Answer (vi):**
To answer this question, we can identify three regions $C$, $D$ and $E$ inside the interferometer. In region $C$ the field is $\frac{\psi_1 + \psi_2}{\sqrt{2}}$. In region $D$ the $\psi_2$ component of the field takes up a phase factor $e^{i\delta}$, the resulting field is,

$$\frac{1}{\sqrt{2}}(\psi_1 + e^{i\delta}\psi_2).$$

In $E$, after the second beamsplitter the field is,

$$\frac{1}{\sqrt{2}} \left( \frac{\psi_1 + \psi_2}{\sqrt{2}} + e^{i\delta} \frac{\psi_1 - \psi_2}{\sqrt{2}} \right).$$

The action of the beamsplitters is unchanged, only that a phase factor has been added. Simplifying the above,

$$\frac{1}{2} \left( \psi_1(1 + e^{i\delta}) + \psi_2(1 - e^{i\delta}) \right) = \frac{e^{i\delta/2}}{2} \left( \psi_1(e^{-i\delta/2} + e^{i\delta/2}) + \psi_2(e^{-i\delta/2} - e^{i\delta/2}) \right)$$

$$= e^{i\delta/2} \left( \frac{\psi_1(e^{-i\delta/2} + e^{i\delta/2})}{2} + \psi_2(e^{-i\delta/2} - e^{i\delta/2}) \right)$$

$$= e^{i\delta/2} \left( \frac{\psi_1(e^{-i\delta/2} + e^{i\delta/2})}{2} - \psi_2(e^{i\delta/2} - e^{-i\delta/2}) \right)$$

$$= e^{i\delta/2} \left( \cos(\delta/2)\psi_1 - i \sin(\delta/2)\psi_2 \right).$$
The coefficient of $\psi_1$ is $e^{i\delta/2} \cos(\delta/2)$ and coefficient of $\psi_2$ is $-ie^{i\delta/2} \sin(\delta/2)$.

$$\text{prob (}D_3\text{ clicking)} = \left|e^{i\delta/2}\cos\delta/2\right|^2$$
$$= e^{-i\delta/2}\cos\delta/2 \cdot e^{i\delta/2}\cos\delta/2$$
$$= \cos^2\delta/2$$

$$\text{prob (}D_4\text{ clicking)} = \left|-ie^{i\delta/2}\sin\delta/2\right|^2$$
$$= ie^{-i\delta/2}\sin\delta/2 \cdot -ie^{i\delta/2}\sin\delta/2$$
$$= \sin^2\delta/2.$$  

Notice that $\text{Prob}(D_3\text{ clicking}) + \text{Prob}(D_4\text{ clicking}) = 1$ as expected.

Yes we can make the detectors click with equal probability by adjusting $\delta$ to $\delta = \pi/2$, then $\text{Prob}(D_3\text{ clicking}) = \text{Prob}(D_4\text{ clicking}) = \cos^2(\pi/4) = \sin^2(\pi/4) = 1/2$

(vii) We now make yet another modification to the apparatus, and arrive at the assembly shown in Fig.(f).

What is the probability of $D_2$, $D_3$ and $D_4$ clicking? Can all of these detectors click together? Can any two click simultaneously? Can some detector never click?

**Answer (vii):**

$D_2$ provides which path information after $B_1$, collapsing the wavefunctions. Hence $D_2$ should click 50% of the times. When $D_2$ does not click, the photon takes the upper
arm and is converted into a superposition $\psi_1 + \psi_2$ get again. But $D_3$ and $D_4$ collapse this wavefunction again, with equal probabilities of each of these detectors clicking. Hence

$$\text{Prob}(D_2 \text{ click}) = 0.5$$
$$\text{Prob}(D_3 \text{ click}) = 0.5 \times 0.5 = 0.25$$
$$\text{Prob}(D_4 \text{ click}) = 0.25.$$

(viii) Draw a sketch of a double slit interference experiment analogous to Fig.(f). Can you mention any analogies between experimenting with beamsplitters and experimenting with the double slit apparatus?

**Answer (viii):**

**ANALOGY:** A beamsplitter creates a superposition, as do the two slits in a double slit apparatus.

(ix) Finally refer to Fig.(g).
Instead of a photon, we now input an electron. In the two arms of the interferometer, we place two tightly focused telescopes $T_1$ and $T_2$ which can “illuminate” and show the presence of the electron. It is also assumed that the act of observing the electron through the telescopes is non-destructive, meaning that the electron is not destroyed upon observation by either $T_1$ or $T_2$. What are the relative probabilities of the detectors $D_3$ and $D_4$ clicking? Explain your reasoning in light of the uncertainty principle.

**Answer (ix):**

If $T_1$ (or $T_2$) clicks, the path of the single electron is determined. We have therefore chosen one path in the realm of possibilities. The momentum of the electron becomes uncertain and interference cannot therefore take place. So $D_3$ and $D_4$ each click with a 50% probability. Note that unlike a photon, the electron is not destroyed. Hence even if $T_2$ (or $T_1$) clicks, the electron can continue its onward journey towards $B_2$.

**Answer (x):**

Can you think how the arrangement in Fig.(d) needs to be modified if we would like to see a delayed choice of the photons “choosing” between interfering or not interfering. I expect you to provide a sketch as an answer to this question.
B₂ is moved in and out after the photon has interacted with B₁. If B₂ is in place, D₃ and D₄ will click with probabilities 1 and 0, while if B₂ is removed, D₃ and D₄ will click with 50% probability.

2. Much evidence indicates that the hydrogen atom is about 0.1 nm in radius. That is, the electron’s orbit (regardless of whether it is circular) extends to about this far from the proton. Accordingly, the uncertainty in the electron’s position is no longer than about 0.1 nm.

(a) Find the uncertainty in momentum. Heisenberg’s uncertainty relationship is

$$\Delta p \Delta r \geq \frac{\hbar}{2}.$$

(b) What is the minimum energy the electron must have if it is confined within a radius of 0.1 nm? Since we know nothing about the electron’s velocity, be careful and use the relativistic expression for the energy. Reason how you choose the momentum of the electron.

(c) What is the theoretical minimum kinetic energy the electron will possess if it is confined this close to the nucleus?

(d) If an electron has this much K.E., can it remain bound with the nucleus? I am expecting a simple calculation to validate your answer.

(Note: An electron in an atom moves in three dimensions; the values in the example apply to the component of its motion along the radial direction.)

Answer 2:
(a) We are given that,

\[ \text{Radius of hydrogen atom} = r = 0.1 \text{ nm} = 0.1 \times 10^{-9} \text{ m} \]

\[ \text{Uncertainty in electron’s position} = \Delta r \sim 0.1 \times 10^{-9} \text{ m}. \]

The minimum uncertainty in momentum can be calculated by using uncertainty relation,

\[ \Delta p \Delta r \geq \frac{\hbar}{2} \]

\[ \Rightarrow \Delta p \sim \frac{\hbar}{2\Delta r} \]

\[ = \frac{1.06 \times 10^{-34} \text{ Js}}{2 \times 0.1 \times 10^{-9} \text{ m}} \]

\[ = 5.3 \times 10^{-25} \text{ kg m s}^{-1}. \]

(b) Let the momentum be at least as big as the uncertainty \( p \sim \Delta p \). Now minimum energy of the electron is,

\[ E = \sqrt{(p_e c)^2 + (m_e c^2)^2} \]

\[ = c \sqrt{p_e^2 + m_e^2 c^2} \]

\[ = 3.0 \times 10^8 \text{ ms}^{-1} \sqrt{(5.3 \times 10^{-25} \text{ kgm}^{-1})^2 + (9.11 \times 10^{-31} \text{ kg})^2(3.0 \times 10^8 \text{ ms}^{-1})^2} \]

\[ = 8.2 \times 10^{-14} \text{ J}. \]

(c) Now the minimum kinetic energy of electron will be the difference of its total and rest mass energy, i.e.,

\[ K = E - m_e c^2 \]

\[ = 8.2 \times 10^{-14} \text{ J} - (9.11 \times 10^{-31} \text{ kg} \times (3.0 \times 10^8 \text{ ms}^{-1})^2) \]

\[ = 1.0 \times 10^{-17} \text{ J} \]

\[ = 62.5 \text{ eV}. \]

(d) The electron will be bound if this kinetic energy is larger in magnitude to the
potential energy of the electron, due to electrostatic attraction, which is,

\[
|V| = \frac{ke^2}{r} = \frac{9 \times 10^9 \text{Nm}^2\text{C}^{-2}(1.6 \times 10^{-19} \text{ C})^2}{0.1 \times 10^{-9} \text{ m}} = 2.3 \times 10^{-18} \text{ J} = 2.3 \times 10^{-18} \text{ eV} \approx 14.4 \text{ eV}.
\]

Since \( K > |V| \), the electron cannot be confined to this small radius. Such a hypothetical atom will be unstable.