Solution Assignment
Uncertainties and data processing

1. If a ray of light passes from air into glass, then the angles of incidence θ_i and reflection θ_r are related by Snell's law,

$$\sin(\theta_i) \; = \; n\sin(\theta_r). \tag{1}$$

The refractive index n can be calculated as,

$$n \; = \; \frac{\sin(\theta_i)}{\sin(\theta_r)}. \tag{2}$$

Suppose the angles are measured as,

$$\theta_i \; = \; (20 \pm 1)^\circ$$
$$\theta_r \; = \; (13 \pm 1)^\circ$$

(a) Find the best estimated value of the refractive index n and uncertainty associated with it.

(b) Calculate the fractional uncertainty in n.

Solution

(a) Given measurands are,

$$\theta_i \; = \; (20 \pm 1)^\circ$$
$$\theta_r \; = \; (13 \pm 1)^\circ$$

The refractive index n is given as,

$$n \; = \; \frac{\sin(\theta_i)}{\sin(\theta_r)} = \frac{\sin(20^\circ)}{\sin(13^\circ)} = 1.52. \tag{3}$$

Using Taylor series approximation, uncertainty in n can be calculated as,

$$\Delta n \; = \; \sqrt{\left(\frac{\partial n}{\partial \theta_i}\Delta\theta_i\right)^2 + \left(\frac{\partial n}{\partial \theta_r}\Delta\theta_r\right)^2}. \tag{4}$$

Differentiating Equation (2) w.r.t θ_i yields,

$$\frac{\partial n}{\partial \theta_i} \; = \; \frac{\cos(\theta_i)}{\sin(\theta_r)} = \frac{\cos(20)}{\sin(13)} = \frac{0.9398}{0.2249} = 4.178,$$

and differentiating w.r.t θ_r gives,

$$\frac{\partial n}{\partial \theta_r} = \frac{-\sin(\theta_i)\cos(\theta_r)}{\sin^2(\theta_r)} = \frac{-\sin(20)\cos(13)}{\sin^2(13)},$$
$$= \frac{-(0.342)(0.974)}{0.0506} = -6.583$$

Substituting the above calculated values in Equation (4) gives,

$$\Delta n = \sqrt{\left(4.178\,(1 \times \frac{\pi}{180})\right)^2 + \left(-6.583\,(1 \times \frac{\pi}{180})\right)^2} = 0.135$$

Finally, the value of n along with its uncertainty can be quoted as,

$$n = (1.5 \pm 0.1).$$

(b) Functional uncertainty can be find out as,

$$\frac{\Delta n}{n} = \frac{0.1}{1.5} = 0.066,$$

and percentage uncertainty is,

$$\frac{\Delta n}{n} \times 100 = \frac{0.1}{1.5} \times 100 = 6.6\% \approx 7\%$$

2. A student studying the motion of a cart on air track measures its position, velocity and acceleration at one instant with the results shown in Table (I).

Variable	Best estimated value	Probable range
Position, x	53.3	53.12 to 53.57 (cm)
Velocity, v	-13.5	-14.0 to -13.0 (cm/s)
Acceleration, a	93	96.7 to 90.4 (cm/s^2)

TABLE I: Measurements of position, velocity and acceleration.

Rewrite these results in the standard form $x_{\text{best}} \pm \Delta x$.

Solution

Position range is given as,

$$[53.12 \quad 53.57]$$

and the uncertainty range can be found out,

$$\Delta x = 53.57 - 53.12 = 0.45 \text{ cm}.$$

The final value of x can be quoted as,

$$x = x_{\text{best}} \pm \Delta x = (53.3 \pm 0.4) \text{ cm}.$$

Range for velocity is given as,

$$[-14.0 \quad -13.0]$$

Uncertainty in v is,

$$\Delta v = -13.0 - (-14.0) = 1 \text{ cm/s}.$$

Hence,

$$v = (-13.0 \pm 1) \text{ cm/s}.$$

Likewise,

$$\Delta a = (96.7 \pm 6) \text{ cm/s}$$

The best estimated value of a can be quoted as,

$$a = (93 \pm 6) \text{ cm/s}^2$$

3. The Van der Waals equation of state for a non-ideal gas is,

$$\left(P + \frac{a}{V_{\text{m}}^2}\right)(V_{\text{m}} - b) = RT, \tag{5}$$

where P is the pressure, V_{m} is the molar volume, T is absolute temperature, R is the universal gas constant with a and b species specific for Van der Waals coefficients and given as,

$$\begin{aligned}
V_{\text{m}} &= (2.000 \pm 0.003) \times 10^{-4}\,\text{m}^3\,\text{mol}^{-1} \\
T &= (298.0 \pm 0.2)\,\text{K} \\
a &= 1.408 \times 10^{-1}\,\text{m}^6\,\text{mol}^{-2}\,\text{Pa} \\
b &= 3.913 \times 10^{-5}\,\text{m}^3\,\text{mol}^{-1} \\
R &= 8.3145\,\text{J}\,\text{K}^{-1}\,\text{mol}^{-1}.
\end{aligned}$$

There is no uncertainty in a, b and R.

Calculate the pressure of the gas and find uncertainty associated with it.

Solution

The Vander Waal's equation is,

$$\left(P + \frac{a}{V_m^2}\right)\left(V_m - b\right) = RT. \tag{6}$$

Rearranging the above equation yields,

$$\begin{aligned}
\left(P + \frac{a}{V_m^2}\right) &= \frac{RT}{(V_m - b)}\,, \\
P &= \left(\frac{RT}{V_m - b} - \frac{a}{V_m^2}\right), \qquad (7)\\
&= \frac{(8.3145)(298)}{2 \times 10^{-4} - 3.913 \times 10^{-5}} - \frac{1.408 \times 10^{-1}}{(2 \times 10^{-4})^2}\,, \\
&= 11882007.83\,\text{Pa}, \\
P &= 11.88 \text{ MPa}.
\end{aligned}$$

Uncertainties in P can be find out using Taylor series approximation,

$$\Delta P = \sqrt{\left(\frac{\partial P}{\partial V_m}\Delta V_m\right)^2 + \left(\frac{\partial P}{\partial T}\Delta T\right)^2}. \tag{8}$$

Differentiating Equation (7) w.r.t V_m,

$$\begin{aligned}
\frac{\partial P}{\partial V_m} &= \frac{-RT}{(V_m - b)^2} + \frac{2a}{V_m^3}\,, \\
&= \frac{-(8.3145)(298)}{(2 \times 10^{-4} - 3.919 \times 10^{-5})^2} + \frac{2(1.408 \times 10^{-1})}{(2 \times 10^{-4})^3}\,, \\
&= -9.58 \times 10^{10} + 3.52 \times 10^{10} = -6.06 \times 10^{10},
\end{aligned}$$

and differentiating w.r.t T yields,

$$\frac{\partial P}{\partial T} = \frac{R}{V_m - b} = \frac{8.3145}{(2 \times 10^{-4} - 3.913 \times 10^{-5})} = 51684.59$$

Plugging above expressions in Equation (8) implies,

$$\begin{aligned}
\Delta P &= \sqrt{(-6.06 \times 10^{10} * 0.003 \times 10^{-4})^2 + (51684.59 \times 0.2)^2}\,, \\
&= 20913.3 = 0.021 \times 10^6\,\text{Pa}\,, \\
&= 0.021\,\text{MPa}.
\end{aligned}$$

Hence the final value of pressure P can be quoted as,

$$P = (11.88 \pm 0.02)\,\text{MPa}.$$

4. Generally, we associate a triangular probability distribution function with analog devices and standard uncertainty can be find out through the second moment of the probability distribution function. Mathematically the triangular probability function $f(x)$ is defined as,

$$\begin{aligned} f(x) &= \frac{(x+a)}{a^2} && \text{for all } x \text{ such that } -a \le x \le 0 \\ f(x) &= \frac{(a-x)}{a^2} && \text{for all } x \text{ such that } 0 \le x \le a \\ f(x) &= 0 && \text{for all other values of } x \end{aligned}$$

Find the mean (first moment) and variance (second moment) of the above given triangular probability distribution function.

Solution:

The mean value of any probability distribution function can be calculated from the first moment, the mathematical expression for which is,

$$< x > = \int_{-\infty}^{\infty} x f(x) dx.$$

For the given triangular probability distribution function, we can write the first moment as,

$$\begin{aligned} &= \int_{-a}^{0} x\left(\frac{x+a}{a^2}\right)dx + \int_{0}^{a} x\left(\frac{a-x}{a^2}\right)dx, \\ &= \int_{-a}^{0} \left(\frac{x^2+ax}{a^2}\right)dx + \int_{0}^{a} \left(\frac{ax-x^2}{a^2}\right)dx, \\ &= \frac{1}{a^2}\left[\frac{x^3}{3}\Big|_{-a}^{0} + \frac{ax^2}{2}\Big|_{-a}^{0} + \frac{ax^2}{2}\Big|_{0}^{a} - \frac{x^3}{3}\Big|_{0}^{a}\right], \\ &= \frac{1}{a^2}\left[\frac{a^3}{3} - \frac{a^3}{2} + \frac{a^3}{2} - \frac{a^3}{3}\right], \end{aligned}$$

yielding,

$$< x > = 0.$$

Hence the mean value (first moment) of a triangular probability distribution function is zero.

The variance can be calculated from the second moment,

$$\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} x^2 f(x)dx, \\
&= \int_{-a}^{0} x^2\left(\frac{x+a}{a^2}\right)dx + \int_{0}^{a} x^2\left(\frac{a-x}{a^2}\right)dx, \\
&= \int_{-a}^{0}\left(\frac{x^3+ax^2}{a^2}\right)dx + \int_{0}^{a}\left(\frac{ax^2-x^3}{a^2}\right)dx, \\
&= \frac{1}{a^2}\left[\frac{x^4}{4}\bigg|_{-a}^{0} + \frac{ax^3}{3}\bigg|_{-a}^{0} + \frac{ax^3}{3}\bigg|_{0}^{a} - \frac{x^4}{4}\bigg|_{0}^{a}\right], \\
&= \frac{1}{a^2}\left[-\frac{a^4}{4} + \frac{a^4}{3} + \frac{a^4}{3} - \frac{a^4}{4}\right], \\
&= \frac{1}{a^2}\left[-\frac{a^4}{2} + \frac{2a^4}{3}\right], \\
&= \frac{1}{a^2}\left[\frac{a^4}{6}\right], \\
&= \frac{a^2}{6}.
\end{aligned}$$

Since the standard uncertainty is the square root of the variance, we can write it as,

$$u = \sigma = \frac{a}{\sqrt{6}}.$$

5. To calibrate a prism spectrometer, a student sends light of 10 different wavelengths through the spectrometer and measures the angle θ by which each beam is deflected. For just the first value of λ, he measures θ,

$$52.5,\ 52.3,\ 52.6,\ 52.5,\ 52.7,\ 52.4,\ 52.2,\ 52.5$$

(a) Calculate the mean and standard deviation.

(b) After several measurements, we can expect about 68 % of the observed values to be within σ_{m}. How many values would you expect to lie within this range ($\bar{\theta} \pm \sigma_{\mathrm{m}}$). How many actually do?

(c) How many values lie within the range of 95 % coverage probability ($\bar{\theta} \pm 2\sigma_{\mathrm{m}}$) and how many lie outside this range?

Solution:

(a) The best estimated value of the angle θ is,

$$< \theta > \; = \; \frac{\sum_{i=1}^{8} \theta_i}{n} = \frac{419.7}{8} = 52.46^\circ \,.$$

Deviations can be found out using the following expression,

$$d_i \; = \; \theta_i - < \theta >,$$

and tabulated in Table (II).

Angle, θ (degree)	Deviations, d_i (degree)	Deviations squared, d_i^2 (degree)^2
52.5	0.0375	1.4×10^{-3}
52.3	-0.1625	2.6×10^{-3}
52.6	0.1375	18.9×10^{-3}
52.5	0.0375	1.4×10^{-3}
52.7	0.2375	56.4×10^{-3}
52.4	-0.0625	3.9×10^{-3}
52.2	-0.2626	68.9×10^{-3}
52.5	0.0375	1.4×10^{-3}

TABLE II: Table for calculated deviations.

The standard deviation is,

$$s \; = \; \sqrt{\frac{\sum_{i=1}^{8} d_i^2}{n}} = \sqrt{\frac{0.1788}{8}} = 0.149^\circ \,,$$

The standard uncertainty is,

$$\sigma \; = \; \sqrt{\frac{n}{n-1}}\,(s) = \sqrt{\frac{8}{7}}\,(0.149) = 0.159^\circ \,,$$

and the standard uncertainty in the mean value is,

$$\sigma_m \; = \; \frac{\sigma}{\sqrt{n}} = \frac{0.159}{\sqrt{8}} = 0.056^\circ \,,$$

The best estimated value can be quoted as,

$$\theta = (52.46 \pm 0.06)^\circ .$$

(b) Type A evaluations have a Gaussian probability distribution function associated with them, shown in Figure (1).

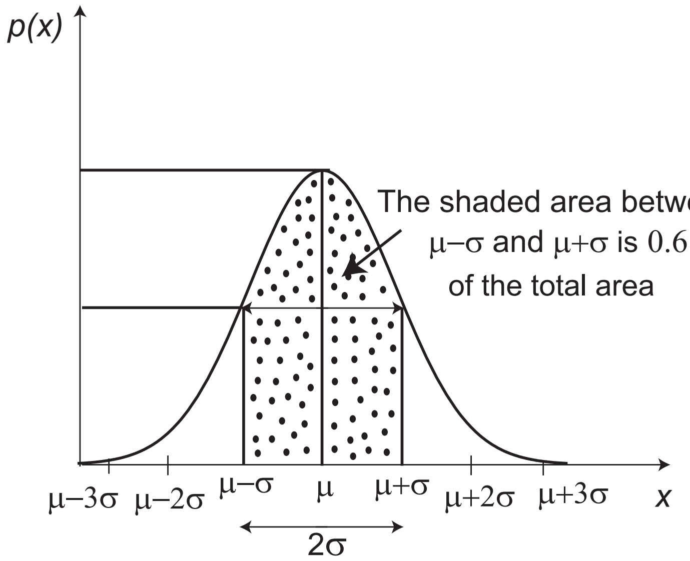

FIG. 1: A Gaussian probability distribution function. Here σ denotes the standard uncertainty in the mean value.

The coverage probability of any arbitrary measurand μ is,

$$(<\mu> \pm 1\sigma) \rightarrow (68\%)$$
$$(<\mu> \pm 2\sigma) \rightarrow (95\%)$$
$$(<\mu> \pm 3\sigma) \rightarrow (99\%)$$

For 68 % confidence interval, the range of the interval becomes,

$$\theta = (52.46 \pm 0.06)^\circ,$$
$$\theta_{\text{range}} = [52.4, 52.52].$$

The values lying within this range are,

$$52.5,\ 52.5,\ 52.4,\ 52.5,$$

and the values lying outside (within 32 %) this confidence of interval are,

$$52.3,\ 52.6,\ 52.7,\ 52.2,$$

(c) For 95 % coverage probability, the range becomes,

$$\theta = (52.46 \pm 2(0.06))^\circ,$$
$$\theta_{\text{range}} = [52.34, 52.58].$$

The values lying within 95 % confidence of interval are,

$$52.5,\ 52.3,\ 52.5,\ 52.4,\ 52.5,$$

and values outside this range are,

$$52.6,\ 52.7,\ 52.2.$$

6. A student measures g using a pendulum of a steel ball suspended by a light string. He records five different lengths of the pendulum and the corresponding time periods T as follows,

Length, l (cm)	51.2	59.7	68.2	79.7	88.3
Time period T (sec)	1.448	1.566	1.669	1.804	1.896

TABLE III: Model table for experimental results.

For each pair, he calculates g through the following expression,

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

(a) Calculate the pair-wise g and find its best estimated value (mean value). Calculate uncertainty in g and quote your final result.

(b) Plot a graph between l and T, find the best estimate of g through the slope of the data given in Table (III).

(c) Which approach do you think yields better result.

Solution:

(a) The value of g can be calculated using the expression,

$$g = \frac{4\pi^2 l}{T^2}.$$

Length, l (cm)	51.2	59.7	68.2	79.7	88.3
Time period T (sec)	1.448	1.566	1.669	1.804	1.896
Acceleration of gravity, g (cm/s^2)	964.03	961.06	966.57	966.82	969.71

TABLE IV: Table for experimental results.

The values of acceleration due to gravity g being calculated through Equation (9) are listed in in Table (IV).

The mean value of g is,

$$< g > \; = \; \frac{\sum_{i=1}^{8} g_i}{n} = \frac{4.828 \times 10^3}{5} = 965.64\,\text{cm/s}^2.$$

Deviations are,

$$d_i \; = \; g_i - < g >,$$

and the outcomes are listed in Table (V).

Acceleration of gravity, g (cm/s^2)	Deviations, d_i (cm/s^2)	Deviations squared, d_i^2 (cm/s^2)^2
964.03	-1.60	2.57
961.06	-4.58	20.96
966.56	0.93	0.86
966.82	1.189	1.39
969.71	4.081	16.61

TABLE V: Table for calculated deviations.

The standard deviation can be calculated as,

$$s \; = \; \sqrt{\frac{\sum_{i=1}^{8} d_i^2}{n}} = \sqrt{\frac{42.3987}{5}} = 2.912\,\text{cm/s}^2\,.$$

The standard uncertainty is,

$$\sigma \; = \; \sqrt{\frac{n}{n-1}}\,(s) = \sqrt{\frac{5}{4}}\,(2.912) = 3.256\,\text{cm/s}^2\,,$$

and the standard uncertainty in the mean value is,

$$\sigma_m = \frac{\sigma}{\sqrt{n}} = \frac{3.256}{\sqrt{5}} = 1.456\,\text{cm/s}^2\,.$$

The final value of g can be quoted as,

$$g = (966 \pm 1)\,\text{cm/s}^2$$

(b) Using the least squares curvefittig, a built-in function of Matlab (lsqcurvefit), we will find the values of slope and intercept. For that, we will follow below given steps:

The independent and dependent row vectors are created as,

≫ **l=[51.2 59.7 68.2 79.7 88.3];**

≫ **T=[1.448 1.566 1.669 1.804 1.896];**

We will plot the data vectors in order to see the relationship between dependent and independent quantities,

≫ **figure ; plot(l, (T.^2), 'ro')**

The model function we will fit on our data would be a linear function, given as,

$$y = mx + c,$$

where m is the slope and c is the intercept.

Now we will make an M-file named **spring** shown in Figure (2a).

Once the fitting function has been defined, we will call the function file in the command window of Matlab, using the following command,

≫ **lsqcurvefit(@spring, [0.05 1], l, (T.^2))**

Now we will redefine the output function by using the values of the parameters that Matlab has returned,

≫ **cfit= 0.0403*l+0.0377;**

Now we plot the redefined function on the data points using the following commands,

≫ **hold on**

≫ **plot(l, cfit)**

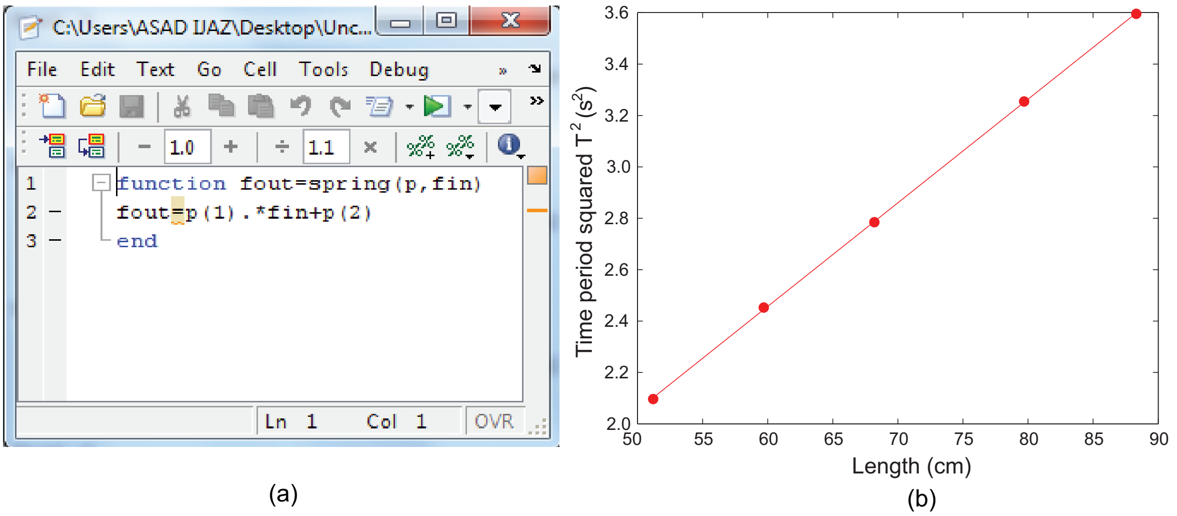

FIG. 2: (a) An M-file for a linear function, (b) Measured data: The initial data points are plotted in red circles while the curve fit is drawn as a solid line.

The final result is depicted in Figure (2b).

The value of the slope that Matlab returns is,

$$\text{slope}\,(m) \;=\; 0.0403, \tag{9}$$

and slope itself equals to the following expression,

$$\text{slope}\,(m) \;=\; \frac{4\pi^2}{g}. \tag{10}$$

Comparing Equations (9) and (10) yields,

$$\frac{4\pi^2}{g} \;=\; 0.0403, \tag{11}$$

implies,

$$g \;=\; 978.62\,\text{cm/s}^2. \tag{12}$$

(c) The method used in part (b) gives g-value which is more closer to the measured value of g ($980\,\text{cm/s}^2$). The significance of least squares curve-fitting is that it minimizes the deviation of each point from the best-fit line and yield better results.

7. A student measures the velocity of a glider on a horizontal air track. He uses a multiflash photograph to find the glider's position s at five equally spaced times as shown in Table (VI).

Time, t (s)	-4	-2	0	2	4
Position s (cm)	13	25	34	42	56

TABLE VI: Experimental data for position and time.

(a) One way to find v would be to calculate ($v = \Delta s/\Delta t$) for each of the four successive two-second intervals and then average them. Show that this procedure gives $v = (s_5 - s_1)/(t_5 - t_1)$, which means that the middle three values are completely ignored by this method. Prove this result.

(b) A better procedure is to make a least squares fit to the equation ($s = s_o + vt$) using all five data points. Follow this procedure to find the best estimate for v and compare your results with that from part (a) (Hint: Use mathematical expressions of least squares fitting of a straight line with equal weights). Find uncertainty in v.

(c) Suppose the time and position is measured through a digital device each with rating of 1 %. Calculated uncertainties in the dependent and independent variables. Plot a graph of the best fit line of the data and show error bars both in the dependent and independent variable.

Solution:

(a) Change in position can be found out by taking the difference of successive intervals,

$$\Delta S = 12,\ 9,\ ,8\ ,\ 14$$
$$\Delta t = 2,\ 2,\ 2,\ 2,$$

The mean values of ΔS and Δt are,

$$< \Delta S > = \frac{\sum_{i=1}^{4}(\Delta S)_i}{n} = \frac{43}{4} = 10.75\,\text{cm},$$
$$< \Delta t > = \frac{\sum_{i=1}^{4}(\Delta t)_i}{n} = \frac{8}{4} = 2\,\text{s},$$

and velocity can be calculted as,

$$v = \frac{\Delta S}{\Delta t} = \frac{10.75}{2} = 5.375\,\text{cm/s}.$$

Now, if we find velocity through the following expression, we get,

$$v = \frac{S_5 - S_1}{t_5 - t_1} = \frac{56 - 13}{4 - (-4)} = \frac{43}{8} = 5.375\,\text{cm/s}.$$

So, by utilizing this method we are only considering the first and the last data points while the middle three values are ignored. Hence, we can conclude that this is not an acceptable approach.

(b) The relationship between time and position is,

$$s = s_o + vt,$$

where v is the slope and s_o is the intercept.

The mean values of the measurands are,

$$\begin{aligned} \bar{s} &= \frac{\sum_{i=1}^{5} s_i}{n} \\ &= \frac{13 + 25 + 34 + 42 + 56}{5} \\ &= 34\,\text{cm}. \end{aligned}$$

$$\begin{aligned} \bar{t} &= \frac{\sum_{i=1}^{5} t_i}{n} \\ &= \frac{-4 - 2 + 0 + 2 + 4}{5} \\ &= 0\,\text{s}. \end{aligned}$$

The best estimated value of the slope can be found out using the relationship,

$$m = \frac{\sum_{i=1}^{N} y_i(x_i - \bar{x})}{\sum_{i=1}^{N} (x_i - \bar{x})^2}.$$

However in our case,

$$m = \frac{\sum_{i=1}^{5} s_i(t_i - \bar{t})}{\sum_{i=1}^{5} (t_i - \bar{t})^2},$$

Implying,

$$\begin{aligned} m &= \frac{[13(-4-0) + 25(-2-0) + 34(0-0) + 42(2-0) + 56(4-0)]}{[(-4-0)^2 + (-2-0)^2 + (0-0)^2 + (2-0)^2 + (4-0)^2]}, \\ &= \frac{206}{40}, \\ &= 5.15\,\text{cm/s}. \end{aligned}$$

Once we find out the slope value, we can use the following relationship for calculating the intercept value,

$$c = \bar{y} - m\bar{x}.$$

Hence,

$$c = \bar{s} - m\bar{t} = 34 - (5.15)(0),$$
$$= 34\,\text{cm}.$$

Uncertainty in slope m is given as,

$$u_m = \sqrt{\frac{\sum_i^N d_i^2}{D(N-2)}},$$

where,

$$d_i = y_i - mx_i - c,$$
$$D = \sum_i^N (x_i - \bar{x})^2.$$

Now,

$$u_m = \sqrt{\frac{(-0.40)^2 + (1.30)^2 + (-0.00)^2 + (-2.30)^2 + (1.40)^2}{3 \times [(-4)^2 + (-2)^2 + (0)^2 + (2)^2 + (4)^2]}},$$
$$= 0.275\,\text{cm/s}.$$

Hence, we can quote our final result as,

$$v = (5.2 \pm 0.3)\,\text{cm/s}.$$

(c) The time t and position s is measured using digital devices and by looking at the data (shown in Table VI), one can tell that the measuring device has a resolution of 1 s for time data and 1 cm for position data. The scale uncertainty can be calculated by associating a uniform probability distribution function with the reading and given as,

$$u_{\text{scale}_{(t)}} = \frac{\Delta}{\sqrt{3}} = \frac{0.5}{\sqrt{3}}\,\text{s}.$$
$$= 0.3\,\text{s}.$$

Since the rating of the device is 1%, uncertainty associated with rating of the instrument is given as,

$$u_{\text{rating}_{(t)}} \text{@}1\% \;=\; 0.01 \times \text{value of } (t),$$

and the combined uncertainty in time t can be calculated using the following expression,

$$u_t \;=\; \sqrt{(u_{\text{scale}_{(t)}})^2 + (u_{\text{rating}_{(t)}})^2}.$$

Likewise for position, the combined uncertainty can be calculated as,

$$\begin{aligned} u_{\text{scale}_{(s)}} &= \frac{\Delta}{\sqrt{3}} = \frac{0.5}{\sqrt{3}}\,\text{cm} = 0.3\,\text{cm}. \\ u_{\text{rating}_{(t)}} \text{@}1\% &= 0.01 \times \text{value of } (s), \\ u_s &= \sqrt{(u_{\text{scale}_{(s)}})^2 + (u_{\text{rating}_{(s)}})^2}. \end{aligned}$$

Calculated values are tabulated in Table (VII).

Time, t (s)	Uncertainty in time, u_t (s)	Position, s (cm)	Uncertainty in position, u_s (cm)
−4.0	0.3	13.0	0.3
2.0	0.3	25.0	0.3
0.0	0.3	34.0	0.3
2.0	0.3	42.0	0.3
4.0	0.3	56.0	0.3

TABLE VII: Experimental data for position and time.

Graph with error bars and best fit line is plotted in Figure (3).

8. If a steel ball is dropped from a certain height into a container of sand, the impact is called a crater. The relationship between the diameter of the crater and the kinetic energy of the impacting object is given as,

$$D \;=\; c\,E^n, \tag{13}$$

where c is a constant, D is the diameter and E is the kinetic energy that can be calculated by assuming that all the kinetic energy possessed by a ball at a height h is converted into potential energy before impact. The data is given in Table (VIII).

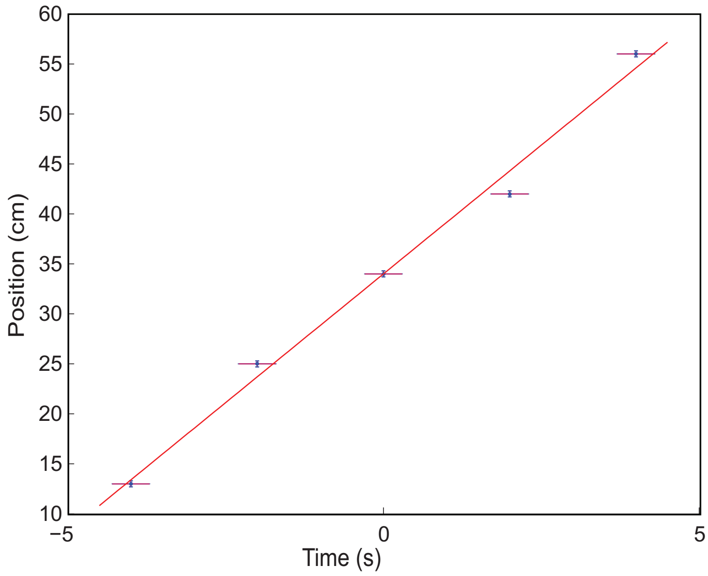

FIG. 3: (a) Graph with error bars both in the independent and dependent variables. The solid red line is the best fit line.

Mass m (g)	**Height h (cm)**	**Crater diameter D (cm)**
8.4	26	$4.0, 4.0, 3.9, 3.9, 4.1, 3.8$
28.2	26	$5.4, 5.3, 5.0, 5.2, 5.3, 5.1$
66.8	26	$6.4, 6.4, 6.2, 6.3, 6.2, 6.3$
66.8	68	$8.2, 7.8, 7.9, 7.9, 8.1, 7.9$
66.8	150	$10.4, 10.0, 10.1, 10.1, 10.2, 10.2$

TABLE VIII: Experimental data for crater formation.

(a) Calculate uncertainties in the independent and dependent variables. The mass is measured using a digital weighing balance (rating= 1%), the height and diameter is measured using a ruler (an analog device).

(b) Calculate uncertainties in $\log(E)$ and $\log(D)$. Plot a graph with error bars, both in the dependent and independent variables.

(c) Using transformation rule, transfer all the uncertainties to the dependent variable. Plot a graph with error bars only in the dependent variable.

(d) Calculate the best estimated values of n and c using weighted fit of a straight line. Calculate uncertainties in n and c as well.

Solution

(a) Equation (13) can be linearized by taking log on both sides,

$$\log(D) \;=\; n\log(E) + \log c, \tag{14}$$

where $(\log D)$ is the dependent variable, $(\log E)$ is the independent variable and the value of n can be calculated by finding the value of slope. This n physically tells about the energy dissipation mechanism.

The mass m of each steel ball is measured using a digital weighing balance and by looking at the data (shown in Table VIII), one can tell that the balance has a resolution of 0.1 g. The scale uncertainty can be calculated by associating a uniform probability distribution function with the reading and given as,

$$u_{\text{scale}} \;=\; \frac{\Delta}{\sqrt{3}} = \frac{0.05}{\sqrt{3}}\,\text{g}. \tag{15}$$

$$\;=\; 0.0289 = 0.03\,\text{g}. \tag{16}$$

Since the rating of the weighing balance is 1%, uncertainty associated with rating of the instrument is given as,

$$u_{\text{rating}}\,@1\% \;=\; 0.01 \times \text{value of } (m), \tag{17}$$

while the combined uncertainty in mass m can be calculated using the following expression,

$$u_m \;=\; \sqrt{(u_{\text{scale}})^2 + (u_{\text{rating}})^2}. \tag{18}$$

The uncertainty in height h has a triangular probability distribution function associated with it and can be calculated using judgement. For the given data, the uncertainty would be,

$$u_h \;=\; \frac{\Delta}{\sqrt{6}} = \frac{0.5}{\sqrt{6}} \;= 0.2041 = 0.20\text{cm}. \tag{19}$$

The type-B uncertainty associated with the diameter of the crater D is,

$$u_D \;=\; \frac{\Delta}{\sqrt{6}} = \frac{0.05}{\sqrt{6}} \;= 0.0408 = 0.04\,\text{cm},$$

since each measurement of the diameter is repeated six times, hence the type-A uncertainty can be found out by calculating the mean, deviation and standard uncertainty in the mean value.

The total uncertainty in the diameter would be the sum of type-A and type-B uncertainties,

$$u_{\text{diameter}} = \sqrt{u_A^2 + u_B^2}.$$

Table (IX) lists the mean values and uncertainties in the diameter.

Crater diameter D (cm)	Mean value $\bar{D}$ (cm)	Type-A uncertainty u_A (cm)	Type-B uncertainty u_B (cm)	Total uncertainty u_{total} (cm)
$4.0, 4.0, 3.9, 3.9, 4.1, 3.8$	3.95	0.04	0.02	0.04
$5.4, 5.3, 5.0, 5.2, 5.3, 5.1$	5.22	0.06	0.02	0.06
$6.4, 6.4, 6.2, 6.3, 6.2, 6.3$	6.30	0.04	0.02	0.04
$8.2, 7.8, 7.9, 7.9, 8.1, 7.9$	7.97	0.06	0.02	0.06
$10.4, 10.0, 10.1, 10.1, 10.2, 10.2$	10.17	0.06	0.02	0.06

TABLE IX: Final outcomes of uncertainties in the diameter

(b) The uncertainty in energy ($E = mgh$) can be calculated using Taylor series approximation,

$$\Delta E = \sqrt{\left(\frac{\partial E}{\partial m}\Delta m\right)^2 + \left(\frac{\partial E}{\partial h}\Delta h\right)^2},$$

Now we need to propagate uncertainties from D and E to $\log(D)$ and $\log(E)$, respectively. This can be found out through the general rule of propagation,

$$u_x = \Delta(\log E) = \sqrt{\left(\frac{\partial(\log E)}{\partial E}\Delta E\right)^2} = \frac{\Delta E}{E},$$

$$u_y = \Delta(\log D) = \sqrt{\left(\frac{\partial(\log D)}{\partial E}\Delta D\right)^2} = \frac{\Delta D}{D}.$$

Graph with error bars both in the dependent and independent variables is shown in Figure (4a).

Uncertainties in all the measured and inferred quantities are quoted in Table (X).

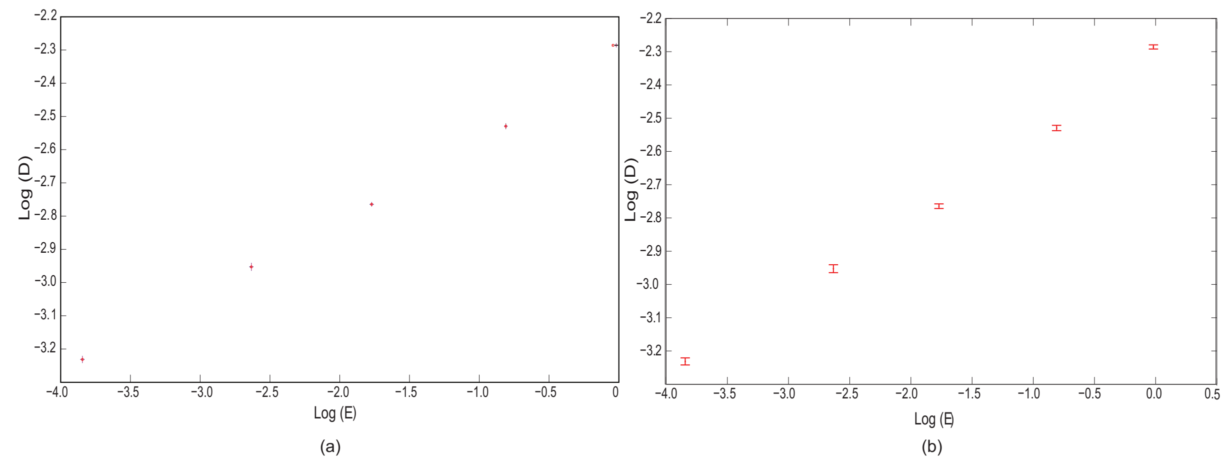

FIG. 4: (a) Graph with error bars both in dependent and independent variable, and (b) graph with error bars only in the dependent variable.

Mass m (g)	Δm (g)	Height h (cm)	Δh (cm)	Energy E (J)	ΔE (J)	Crater diameter D ($\times 10^{-2}$) (m)	ΔD $\times 10^{-2}$ m	$\log(E)$	$\Delta(\log E)$	$\log(D)$	$\Delta(\log D)$
8.4	0.1	26.0	0.2	0.0214	0.0003	3.90	0.04	-3.844	0.013	-3.232	0.010
28.2	0.3	26.0	0.2	0.0719	0.0009	5.20	0.06	-2.633	0.013	-2.953	0.012
66.8	0.7	26.0	0.2	0.170	0.002	6.30	0.04	-1.771	0.013	-2.765	0.006
66.8	0.7	68.0	0.2	0.445	0.005	7.90	0.06	-0.809	0.010	-2.530	0.008
66.8	0.7	150.0	0.2	0.982	0.010	10.20	0.06	-0.018	0.010	-2.286	0.006

TABLE X: Experimental data and calculated uncertainties for crater formation.

(c) The mathematical expressions for transferring uncertainties to the dependent variable ($\log D$) and for calculating the total uncertainty are,

$$u_{\text{Trans}} = \left(\frac{dy}{dx}\right) u_x = (0.24)\, u_x, \tag{20}$$

$$u_{\text{Total}} = \sqrt{(u_{\text{Trans}})^2 + u_y^2}. \tag{21}$$

The uncertainties calculated for the given data of independent ($\log E$) and dependent variables ($\log D$) are shown in Table (XI).

Graph is shown in Figure (4b).

(d) The weights w are reciprocal squares of the total uncertainty being utilized in

log(E)	log(D)	u_{Trans}	u_{Total}
-3.844	-3.244	0.004	0.011
-2.633	-2.956	0.004	0.009
-1.771	-2.765	0.004	0.008
-0.809	-2.538	0.002	0.005
-0.018	-2.283	0.002	0.004

TABLE XI: Transferred and total uncertainties.

least-squares fitting of a straight line. The expression for calculating the weight w is,

$$w = \frac{1}{u_{\text{Total}}^2}\,. \tag{22}$$

The mathematical relationships for slope (m) and intercept (c) are,

$$m = \frac{\Sigma_i w_i\,\Sigma_i w_i(x_i y_i) - \Sigma_i(w_i x_i)\,\Sigma_i(w_i y_i)}{\Sigma_i w_i\,\Sigma_i(w x_i^2) - (\Sigma_i w_i x_i)^2}\,, \tag{23}$$

and,

$$c = \frac{\Sigma_i(w_i x_i^2)\,\Sigma_i(w_i y_i) - \Sigma_i(w_i x_i)\,\Sigma_i(w_i x_i y_i)}{\Sigma_i w_i\,\Sigma_i(w_i x_i^2) - (\Sigma_i w_i x_i)^2}, \tag{24}$$

where x is the independent variable ($\log E$ in our case), y is the dependent variable ($\log D$) and w is the weight.
The numerator and denominator of Equations (23) and (24) are calculated separately and tabulated in Table (XII).

Substituting the calculated terms given in Table (XII) in Equation (23) yields,

$$\begin{aligned} m &= \frac{(76563)(297500) - (-101750)(-201640)}{(76563)(253900) - (1.0352\times 10^{10})}, \\ &= 0.2488 \end{aligned}$$

and the intercept (c) is given as,

$$\begin{aligned} c &= \frac{(253900)(-201640) - (-101750)(297500)}{(76563)(253900) - (1.0352\times 10^{10})}, \\ &= -2.3027 \end{aligned}$$

w	wxy	wx	wy	wx^2
0.8886×10^4	1.1038×10^5	-3.4159×10^4	-2.8714×10^4	1.3132×10^5
0.7068×10^4	0.5495×10^5	-1.8610×10^4	-2.0869×10^4	0.4900×10^5
2.0148×10^4	0.9863×10^5	-3.5676×10^4	-5.5701×10^4	0.6317×10^5
1.5882×10^4	0.3251×10^5	-1.2854×10^4	-4.0173×10^4	0.1040×10^5
2.4580×10^4	0.0102×10^5	-0.0447×10^4	-5.6183×10^4	0.0001×10^5
$\sum w = 76563$	$\sum wxy = 297500$	$\sum wx = -101750$	$\sum wy = -201640$	$\sum wx^2 = 253900$

TABLE XII: Terms being utilized in weighted fitting of a straight line.

The mathematical expressions for finding the uncertainties in slope (m) and intercept (c) are,

$$u_m = \sqrt{\frac{\Sigma_i w_i}{\Sigma_i w_i \, \Sigma_i (w_i x_i^2) - (\Sigma_i w_i x_i)^2}}, \tag{25}$$

$$u_c = \sqrt{\frac{\Sigma_i (w_i x_i^2)}{\Sigma_i w_i \, \Sigma_i (w_i x_i^2) - (\Sigma_i w_i x_i)^2}}. \tag{26}$$

Substituting the values from Table (XII) in the above given expressions yield,

$$u_m = \sqrt{\frac{(76563)}{(76563)(253900) - (1.0352 \times 10^{10})}} = 0.0029, \tag{27}$$

$$u_c = \sqrt{\frac{(253900)}{(76563)(253900) - (1.0352 \times 10^{10})}} = 0.0053. \tag{28}$$

Finally, the values of slope and intercept alongwith uncertainties are quoted as,

$$m = (0.249 \pm 0.003). \tag{29}$$

$$c = (-2.303 \pm 0.005). \tag{30}$$

Since the value of slope is the n value which tells about the energy dissipation mechanism , hence we can conclude,

$$n = (0.249 \pm 0.003). \tag{31}$$