Lab Monograph

Experimental Physics Laboratory PH331


Muhammad Sabieh Anwar
Contents

1 Uncertainties in Measurements ............................................. 1
   1.1 Rate of evaporation of water ........................................... 1
   1.2 Craters in sand .......................................................... 2
   1.3 Coefficient of static friction .......................................... 4
   1.4 Density of a metal .................................................... 5

2 Primer on the Lock-in Amplifier ......................................... 8
   2.1 List of Equipment ..................................................... 8
   2.2 Objectives ............................................................. 9
   2.3 Theoretical Introduction ............................................ 10
   2.4 Finding a weak optical signal buried in noise .................... 15
   2.5 Finding an unknown resistance ..................................... 17

3 Temperature oscillations in a metal: Probing aspects of Fourier analysis ......................................................... 19
   3.1 Objectives ............................................................ 20
   3.2 Theoretical background ............................................. 21
   3.3 The experiment ...................................................... 26

4 Principles and Applications of Superconducting Quantum Interference Devices ....................................................... 29
4.1 Objectives ........................................ 30
4.2 Theoretical Introduction .......................... 31
4.3 Overview of the Mr. SQUID apparatus .......... 46
4.4 Experiments ........................................ 49

5  Chasing Chaos with an RL-Diode Circuit ........ 68
  5.1 Objectives ........................................ 69
  5.2 Foundations ....................................... 70
  5.3 Identifying Chaos .................................. 73
  5.4 The Experiment .................................... 80

6  Phase Sensitive Faraday Rotation .................. 86
  6.1 Objectives ........................................ 87
  6.2 Theoretical introduction .......................... 89
  6.3 Experimental Technique .......................... 97
  6.4 The Experiment .................................... 108
  6.5 (OPTIONAL) Measurement of the the Verdet constant using higher harmonic components ............ 114

7  Studying Phase Transitions with a Strain Gage .......... 117
  7.1 Objectives ........................................ 118
  7.2 Introduction ..................................... 120
  7.3 Overview of the Experiment ....................... 123
  7.4 Experimental Procedure .......................... 128
  7.5 Experimental Objectives .......................... 129

8  Statistical properties of White Noise (Electronics & Signal Processing) .................. 130
## CONTENTS

8.1 Experimental Objectives ........................................ 130
8.2 Filtering White Noise ........................................... 132

9 Michelson Interferometry ........................................ 138

9.1 List of Equipment ................................................... 138
9.2 Experimental Objectives .......................................... 139
9.3 Introduction ....................................................... 140
9.4 Optical alignment of the Michelson interferometer .......... 141
9.5 Measuring wavelength of the laser using Michelson Interferometer 142
9.6 Measuring refractive index of glass ................................ 143
9.7 Appendix A .......................................................... 144

10 Band Structure and Electrical Conductivity in Semiconductors 146

10.1 Objectives ......................................................... 147
10.2 Theoretical introduction ......................................... 148
10.3 The experiment ................................................... 160
10.4 How to construct a temperature profile in temperature controller 171
10.5 PID controller .................................................... 173

11 Tracking Brownian Motion through Video Microscopy 175

11.1 Objectives ........................................................ 175
11.2 Theoretical Introduction ......................................... 178
11.3 Apparatus .......................................................... 181
11.4 Experimental method ............................................. 183
11.5 Precautions in microscope handling .............................. 194

12 The Magnetic Pendulum ........................................... 195
12.1 Objectives ................................................. 196
12.2 Introduction ............................................. 197
12.3 Apparatus ................................................. 198
12.4 The Experiment ........................................ 200

13 Synthesis and Ferroelectric Properties of KNO₃ films............. 204

13.1 List of Equipment ....................................... 205
13.2 Objectives ............................................... 206
13.3 Theoretical background .................................. 206
13.4 The Experiment ......................................... 212
Chapter 1

Uncertainties in Measurements

Amrozia Shaheen, Asma Khalid and Muhammad Sabieh
Anwar

This is an outline of some very simple experiments whose sole purpose is to get
familiarized with the concept of uncertainty and how it can be quantified and
expressed.

1.1 Rate of evaporation of water

Experimental method

Take a small beaker, measure its diameter using the provided vernier callipers.
Repeat the measurements for six values of the diameter. Then add water into a
beaker, such that the level of the water is around 2 mm from the bottom. Place
the beaker on an electronic weigh balance. Use a stop watch to measure the mass
of the water at intervals of 300 s. Continue taking readings for approximately
3600 s. Measure and record the room temperature.
Figure 1.1: (a) Schematic diagram of the apparatus for measuring the rate of evaporation of water, and its (b) photograph.

Objectives

1. Calculate the best value of the diameter by considering random and systematic errors, also find the combined standard uncertainty in the diameter.

2. Evaluate the effective number of degrees of freedom, \( \nu_{\text{eff}} \), for the combined standard uncertainty in the diameter.

3. Plot the mass versus time graph and calculate the best estimate of slope as well as standard deviation of the slope.

4. Determine the best estimated value and the combined uncertainty in the rate of evaporation of water per unit area.

5. Find out the effective degrees of freedom for the combined standard uncertainty in the evaporation rate and find the coverage factor at the 95\% level of confidence.

6. Calculate the true value of the evaporation rate at the 95\% level of confidence.

1.2 Craters in sand

Experimental method

Setup the apparatus as shown in Figure (1.3). Find the masses of the available steel ball bearings. Level up the sand by shaking the container vigorously. Drop the balls, one by one, into the sand container from a range of different heights.
1.2. CRATERS IN SAND

between 25 cm to 95 cm. The heights are measured from the surface of the sand. Find the diameter $D$ of the crater using a ruler, as shown in Figure (1.3). Take at least 3 replicate measurements of diameter at a specific height, so that we can find the mean and spread of the data for a specific height and mass of the ball. Shake the container vigorously after taking every reading. This is to ensure that the sand is not compacted. Repeat the above process for different balls to get a range of kinetic energies of the impacting object.

![Clamp, Stand with meter, ruler attached, Ball bearings of different masses, Sand]

Figure 1.2: Photograph of the crater formation experiment.

![Cross section of the sand crater formed by falling ball into the sand container. The position of the peak of the crater wall is used to estimate its diameter.]

Figure 1.3: Cross section of the sand crater formed by falling ball into the sand container. The position of the peak of the crater wall is used to estimate its diameter.

**Objectives**

1. The diameter of the crater, $D$, and the kinetic energy, $E$, of the impacting ball are related through the relation $D = cE^n$.

2. Determine the best value and the combined standard uncertainty in the
kinetic energy, $E$, taking into account the uncertainties in the mass, acceleration due to gravity and the height of the ball.

3. Calculate the estimated value and the combined uncertainty in the diameter.

4. Our goal is to find $n$ which will specify the mechanism of crater formation [1]. For example, $n = 1/3$ implies that the dominant mechanism is the plastic deformation of the sand surface and $n = 1/4$ suggests that the craters are formed by the ejection of sand. In order to estimate $n$ relevant to our experiment, plot a graph of $\ln D$ versus $\ln E$. Evaluate the best estimate of the slope, also find the standard deviation in the slope.

5. Calculate the number of degrees of freedom and use it to determine the coverage factor at the 95% level of confidence.

6. Evaluate the expanded uncertainty and the true value of the slope at the 95% level of confidence.

### 1.3 Coefficient of static friction

**Experimental method**

Place a glass slide on a glass block. Move the glass block slowly at an angle of inclination, until critical angle is reached and the glass slide began to slip as shown in Figure (1.4). Use a protractor of smallest division of $1^\circ$ to measure the critical angle. Return back the glass block and glass slide to their original positions and repeat the procedure at least for six replicate values of critical angle. Once again, repeat the above steps for a wood piece placed on a glass block.

![Figure 1.4: A glass slides on an inclined block of glass. $\theta_c$ is the critical angle.](image-url)
Objectives

1. Calculate the best estimated value of the critical angle, \( \theta_c \), taking into account the random and systematic errors and combined standard uncertainty.

2. Find out the effective number of degrees of freedom for the combined standard error in the critical angle, \( \nu_{\text{eff}} \), using Welch-Satterthwaite formula.

3. Our aim is to find out the coefficient of static friction, for which we will use the relationship \( \mu_s = \tan(\theta_c) \). Calculate the best estimated value of the coefficient of static friction for glass on glass and wood on glass. The relationship \( \mu_s = \tan(\theta_c) \) is derived from the formula, \( F_{s,\text{max}} = \mu_s N \) where \( F_{s,\text{max}} \) is the force of static friction and \( N \) is the normal force exerted by the inclined plane on the sliding object.

4. Find out the coverage factor from the calculated effective number of degrees of freedom and determine the expanded uncertainty for the true value of the coefficient of static friction at the 95\% level of confidence.

1.4 Density of a metal

Experimental method

Place the metal ball bearing on an electronic weight balance of resolution 0.01 g. Note down the mass of the ball and repeat the measurements for a set of eight readings. Measure the diameter of the steel ball bearing using a screw gauge of resolution 0.01 mm. Take a set of six replicate measurements.

![Figure 1.5: Measuring the mass of metal balls.](image-url)
CHAPTER 1. UNCERTAINTIES IN MEASUREMENTS

Objectives

1. Calculate the best estimated value and the combined standard uncertainty in mass and diameter.

2. Evaluate the effective number of degrees of freedom, $\nu_{\text{eff}}$, for the combined standard uncertainty in mass and diameter using the Welch-Satterthwaite formula.

3. Evaluate the standard uncertainty and the effective number of degrees of freedom for the density of the ball bearing. Evaluate the coverage factor at the 95% level of confidence.

4. Determine the true value of the density by evaluating the expanded uncertainty at the 95% level of confidence.
Bibliography


Chapter 2

Primer on the Lock-in Amplifier

Sidra Farooq, Rabiya Salman, Wasif Zia, Umer Hassan and Muhammad Sabieh Anwar

The resistance of a Cu wire is so small that it cannot be accurately and reliably measured by ordinary voltmeter. The same is true for a gold nanowire tethered between two supports. In experiments, we repeatedly come across situations where we want to measure the response of a system and the response is feeble or is buried in excessive noise. Through this experiment we introduce the art of precise measurements using a lock-in amplifier which is an integral component of several experiments in the physics lab. As such, the present experiment is a pre-requisite to some of the advanced experiments in this course.

KEYWORDS

Lock-in amplifier · White noise · Flicker noise · Thermal noise · Phase sensitive detection · Low-pass filter · Optical chopper

SCHEDULED TIME 1 WEEK

2.1 List of Equipment

1. Optical Chopper, SR540, Stanford Research Systems (SRS)
2.2 Objectives

In this experiment, we will,

1. eliminate unwanted noise, extract signals buried in excessive noise using lock-in amplifier.

2. learn about different types of noise, and see how noise can obscure the required signal and how phase-sensitive techniques can be used to recover the signal.

3. operate optical chopper to measure weak optical signals modulated at a certain frequency.

4. find the minute resistance of a conducting wire.
Bibliography


2.3 Theoretical Introduction

Noise and its types

Noise include all those voltages and currents that accompany and obscure the signal of interest. Noise is generally classified into two kinds: white noise and pink noise [2].

White noise has all frequency components, ranging from zero frequency (DC) to infinite frequencies. The graph between the noise power density versus frequency would be a constant line. Thermal or Johnson noise is a type of white noise. It arises from the thermal fluctuations in a resistor at finite temperature. The rms amplitude of thermal noise voltage is:

$$V_{\text{rms, thermal}} = \sqrt{4k_BTRB},$$

where $R$ is the resistance and $B$ is the bandwidth of measurement, $T$ is the absolute temperature, and $k_B$ is the Boltzmann Constant.
2.3. THEORETICAL INTRODUCTION

Shot noise is another type of white noise which arises due to fluctuation of single charge carriers, e.g., electron flowing through a single electron transistor or a highly attenuated beam of practically single photons impinging on a photodetector.

Pink noise has a power spectral density that decreases with frequency. It has a frequency dependence of $1/f$. It is sometimes also referred to as flicker noise.

"Intrinsic" noise sources include resistors, vacuum diodes, p-n junctions etc. In addition to the intrinsic noise sources, there are a variety of "non-essential" noise sources, i.e., those noise sources which can be minimized with good laboratory practices. Some of these extrinsic noise sources are highlighted below.

**Capacitive Coupling** A voltage on a nearby piece of apparatus can couple to a detector via stray capacitance. Although the stray capacitance maybe small, but when coupled in, the resultant noise may be larger than a weak experimental signal.

![Capacitive Noise Coupling](image)

Figure 2.1: Capacitive Noise Coupling.

It can be cured by removing or turning off the interfering noise source, or by installing capacitive shielding which can be done by placing the setup in a metal box. Consider Figure 2.1.

**Inductive Coupling** Extrinsic noise can also couple to the experiment via a magnetic field. A changing current in a nearby circuit gives rise to a changing magnetic field which induces an emf in the loop connecting the detector to the experiment. See Figure 2.2. This can be cured by using a magnetic shielding, or by using twisted pair cables. An example of inductive coupling is the noise due to 50 Hz main power line, called "line interference" or 50 Hz hum. It can be reduced by placing a notch filter which is centered around the line frequency.

**Resistive Coupling** Currents through common ground connections can give rise to noise voltages. This arises when we have two different grounding points which are not at exactly the same potential. Thus, the detector measures the voltage...
across the experiment plus the voltage due to noise current passing through the finite resistance of the ground bus. This is shown by \( V_G \) in Figure 2.3. Resistive coupling can be cured by grounding everything to the same physical point.

Other noise sources include mechanical vibrations.

Laboratory practices can reduce noise sources to a manageable level and the lock-in amplifier employing the technique of phase sensitive detection can be used to recover signals which may still be buried in noise.

**Internal working of the Lock-in Amplifier**

The lock-in amplifier is used to detect and measure very small AC signals. A lock-in amplifier can make accurate measurements of small signals even when the signals are obscured by noise sources which may be thousand of times larger. Essentially, a lock-in is a filter with an extremely narrow bandwidth which is tuned to the frequency of the signal. Such a filter will reject most unwanted noise to allow the signal of specific frequency to be measured. For complete details the reader is referred to [3]. Figure 2.4 shows the approximate internal schematic of a typical lock-in amplifier.

The lock-in amplifier is basically a phase sensitive detector: a mixer followed by a low pass filter. These components are explained here.
Mixer

The lock-in technique requires a reference frequency. We synchronize or derive the signal of interest from a suitable reference signal. PSD operates by multiplying two signals together.

The demodulation process is analyzed in two ways: graphically as well as mathematically. In the graphical method, illustrated in Figure (2.5), different possibilities for the signal are shown. The left most signal (a) is in phase with the reference signal and the right most (c) is 180° out of phase with the reference, while the center one (b) is 90° out of phase. First, consider the case (a). During first half cycle the reference is positive, the mixer output is positive. In the negative half cycle of the reference, the mixer output is again positive. Therefore, when the reference and sinusoids are in phase, the mixer output is a full wave rectified sinusoid, whose dc component is proportional to the input signal.

★ Q1. Explain the output when the input signal is 90° and 180° out of phase with the reference signal.

We can also describe the mixing operation mathematically. Consider a sinusoidal input signal,

\[ V_{in} = A \sin \omega t \quad \text{with} \quad \omega = 2\pi f. \]  

(2.1)

where \( f \) and \( \omega \) are the angular frequencies of the signal. A square wave can be represented as the sum of odd harmonics of the sinusoid. Suppose we have a reference signal of amplitude \( B \), frequency \( \Omega \) at phase \( \phi \) relative to the input
Figure 2.5: The output $V_o$ is determined by multiplying the signal and the reference wave.

signal and considering the fundamental only,

$$V_{ref} = B \sin(\Omega t + \phi).$$  \hspace{1cm} (2.2)

The mixer operates by multiplying the two signals together so, the output $V_o$ will be,

$$V_o = A \sin(\omega t) B \sin(\Omega t + \phi) \hspace{1cm} (2.3)$$

$$= \frac{AB}{2} (\cos((\omega - \Omega)t + \phi) - \cos((\omega + \Omega)t + \phi)) \hspace{1cm} (2.4)$$

showing that the mixer output comprises two AC signals, one at the difference frequency $(\omega - \Omega)$ and other at the sum frequency $(\omega + \Omega)$. If reference frequency is equal to the frequency of input signal i.e. $\omega = \Omega$, a sinusoidal output is obtained with some DC offset. Figure (2.5).

$$V_o = \frac{AB}{2} (\cos(\phi) - \cos(2\Omega t + \phi)).$$  \hspace{1cm} (2.5)

So, the output $V_o$ is proportional to the magnitude of input signal $A$, the cosine of angle between input and reference and it is modulated at twice the reference frequency.

**Low Pass Filter**

If the output voltage from the mixer, $V_o$ is passed through a low pass filter whose cut off is $\ll 2\Omega$, the sinusoidal component is removed and we are left with the
DC part only.

\[ V_o = \frac{AB}{2} \cos(\phi). \]  

(2.6)

Noise close to reference frequency contributes to a small magnitude to mixer output. The noise rejection depends upon the low pass filter bandwidth and the roll-off. As the bandwidth is made narrower, the noise rejection is improved.

### 2.4 Finding a weak optical signal buried in noise

**Objective**

In this section, students will detect a weak optical signal using an optical chopper and a lock-in amplifier.

When two polarizers are placed nearly crossed with each other, we get a very small signal on the photodetector which cannot be measured by an ordinary voltmeter. The lock-in amplifier can be used for the detection of such small signals, which are close to but not quite zero. This extremely small signal will otherwise be washed out by ambient light.

The signal coming from the continuous wave laser is a DC signal whereas we require a modulated signal for the lock-in to operate. For this purpose, we use an optical chopper.

**Optical Chopper**

Modulation is the key step that enables the use of the lock-in amplifier. Modulation is the conversion of the DC signal to an AC signal with a defined frequency. In the present experiment, an optical chopper is used to square wave modulate the intensity of an optical signal. It comprises a chopping wheel, a motor and a speed control mechanism. The chopping wheel/blade is a rotating metallic disk with slots cut in the blade. The speed and number of slots determine the modulating frequency of the chopper. The modulation not only differentiates against noise but also discriminates against background light of constant intensity. Our optical chopper, SR540 SRS, also provides an output frequency equal to the modulation frequency, which serves as the reference frequency for the lock-in amplifier.
**Procedure**

Carry out the following procedure to measure a weak signal (obtained by crossing two polarizers), using the lock-in amplifier and optical chopper.

![Diagram of weak signal measurement](image)

Figure 2.6: Weak Signal Measurement.

1. Connect the setup as shown in the Figure 2.6.

2. Switch on the optical chopper (study [4] before operating). Set the frequency to around 230 Hz. Keep the polarizers nearly crossed. The reference signal for the lock-in amplifier is provided by the chopper.

   ★ **Precaution** Always check the “OVL” blinking on the lock-in amplifier. If it does, adjust the ‘sensitivity’ button.

   ★ **Q 2.** Connect the photodiode’s output to the provided I/V convertor. Observe the I/V’s output and the reference signal on the oscilloscope operating in the ‘dual mode’. Describe your observations.

   ★ **Q 3.** Adjust the polarizers to bring about a peak output voltage of 100 mV from the photodiode, as observed on the oscilloscope. Then, connect the signal to the lock-in amplifier. Adjust the sensitivity, pre and post filters time constants and find the output. The lock-in amplifier shows an rms value in volts.

   ★ **Q 4.** Find the Fourier series of the I/V’s output of Q2 and find the coefficient of the fundamental frequency. Correlate it with the measurement on the lock-in amplifier of Q3.
2.5 Finding an unknown resistance

★ Q 5. Calculate the current being produced by the photodiode using the result of Q3.

★ Q 6. Now, connect the photodiode’s output to the current input of the lock-in amplifier. Use the $10^6$ V/A conversion factor to find the current. Verify the two readings.

★ Q 7. Now, cross the polarizers rendering the output on the oscilloscope immeasurable. Then, connect the output to the lock-in amplifier and see if you can still measure the output voltage. This step should be the most revealing aspect of the lock-in amplifier: its ability to detect a feeble voltage which is otherwise immeasurably small. Fully describe your observations and the settings of the lock-in amplifier.

★ Q 8. Find the output of the lock-in amplifier when the reference mode is adjusted to $2f$. What does the output represent?

★ Q 9. In Q7, what is the optical power of the laser light after passing through the crossed polarizer? You may need to consult the responsivity curve of the photodetector [5].

★ Q 10. Find the output voltage using lock-in amplifier at the chopping frequency of 100 Hz. Can you explain the cause of the strange reading at this frequency?

2.5 Finding an unknown resistance

There are essentially two ways to measure the electrical resistance of any device: sending in a known current and measuring the resulting voltage across it or applying a known voltage and measuring the resulting current through it. Technically the former option is easier.

★ Q 11. Find the theoretical value of $R$ for a lead-tin alloy of length $L$ and diameter $d$. The lead-tin alloy is the commonly used alloy in jumper wires. Resistivity of the tin alloy is approximately 0.109 $\mu\Omega m$.

★ Q 12. Set up the apparatus according to Figure 2.7. Generate a 1 Volt, 80 Hz sine wave using the signal generator. Now using the lock-in amplifier, find $V_{\text{wire}}$.

★ Q 13. Calculate $R_{\text{wire}}$ using the following expression,
Figure 2.7: Circuit Diagram.

\[ R_{\text{wire}} = \frac{R_{\text{series}} V_{\text{wire}}}{V_{\text{in}} - V_{\text{wire}}} \]  \hspace{1cm} (2.7)

★ Q 14. Derive the above expression using the approximation \( R_{\text{series}} \gg R_{\text{wire}} \).

★ Q 15. Compare the theoretical and experimental values of \( R_{\text{wire}} \).

★ Q 16. Measure \( R_{\text{wire}} \) at very low frequencies. Explain the results. Hint: The measurements are much noisier at low frequencies because of \( 1/f \) noise.
Chapter 3

Temperature oscillations in a metal: Probing aspects of Fourier analysis

Sohaib Shamim, Wasif Zia and Sabieh Anwar

The purpose of this experiment is to acquaint you with physical illustration of concepts in Fourier analysis all using a simple experimental setup involving wave-like behavior. In addition, you will measure the speed of propagation of thermal oscillations, analyze the heat equation and measure the thermal diffusivity of the material under observation. The temperature oscillations are distinct from true "travelling waves" as they do not transfer energy, and arise out of a diffusive or scattering process. But the concept of a "wave", as simple, innocuous and ubiquitous it seems, is exceedingly difficult and multi-faceted to the extent that no universal definition is possible [1]!

KEYWORDS

Heat equation · Fourier series · Fourier transform · Harmonics · Damping · Diffusivity.

APPROXIMATE PERFORMANCE TIME 1 week.
3.1 Objectives

In this experiment, we will,

1. understand the basis of heat flow and recognize heat conduction as a diffusive process,
2. learn about solutions of the heat equation,
3. decompose an oscillation into its harmonics,
4. observe different harmonics and how they damp with different rates, and
5. estimate the thermal diffusivity of a metal.
Bibliography


3.2 Theoretical background

Heat equation

Experiments have shown that heat flow is proportional to the gradient of the temperature. If the heat flux is \( \vec{J} \), then

\[
\vec{J} = -\kappa \nabla T. 
\]

(3.1)

where \( T(x, y, z, t) \) is the temperature, \( t \) is time and \( \kappa \) is defined as the thermal conductivity.

The heat flowing out of a volume, \( V \), bounded by \( S \), is,

\[
\int_S \vec{J} \cdot \hat{n} dS = \int_V \nabla \cdot \vec{J} dV. 
\]

(3.2)

Now the total thermal energy inside \( V \) is,
\[ \int_V C T dV = \int_V \sigma \rho T dV. \]  

(3.3)

with \( C \) being the specific heat capacity (\( JK^{-1} \text{mol}^{-1} \)), \( \sigma \) the per unit mass heat capacity (\( JK^{-1} \text{kg}^{-1} \)) and \( \rho \) the mass density.

The rate of loss of energy through \( S \) is,

\[ -\frac{\partial}{\partial t} \int_V \sigma \rho T dV = -\sigma \rho \int_V \frac{\partial T}{\partial t} dV. \]  

(3.4)

Equating (3.2) and (3.4) we obtain,

\[ \nabla \cdot J = -\sigma \rho \frac{\partial T}{\partial t}. \]  

(3.5)

and substituting (3.1) into (3.5).

\[ \nabla^2 T = \frac{\sigma \rho}{\kappa} \frac{\partial T}{\partial t} \]  

(3.6)

\[ = \frac{1}{D} \frac{\partial T}{\partial t}. \]  

(3.7)

where \( D = \kappa/\sigma \rho \) is called the diffusivity of the material.

**Q 1.** How does the heat equation compare with (a) the wave equation \( \nabla^2 \psi = \frac{1}{\sigma^2} \frac{\partial \psi}{\partial t} \).

(b) the time dependent Schrodinger Equation?

**Q 2.** Show that for heat flow along a one-dimensional wire or rod, the heat equation becomes,

\[ \frac{\partial^2 T}{\partial x^2} = \frac{1}{D} \frac{\partial T}{\partial t}. \]  

(3.8)

**Q 3.** Notice the analogies in the following table.

<table>
<thead>
<tr>
<th>Conduction of heat</th>
<th>Diffusion of particles</th>
<th>Electric current</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \vec{J} = -\kappa \nabla T )</td>
<td>( \vec{\Phi} = -D \nabla n )</td>
<td>( \vec{J} = -\sigma \nabla \phi )</td>
</tr>
<tr>
<td>( \kappa ): thermal conductivity</td>
<td>( D ): diffusion constant</td>
<td>( \sigma ): electrical conductivity</td>
</tr>
<tr>
<td>( T ): temperature</td>
<td>( n ): concentration of particles</td>
<td>( \phi ): electric potential</td>
</tr>
<tr>
<td>( \vec{J} ): heat flow rate</td>
<td>( \vec{\Phi} ): flow rate of particles</td>
<td>( \vec{J} ): current density</td>
</tr>
</tbody>
</table>

Table 3.1: The analogies between apparently different physical phenomena, all unified through the heat equation.
3.2. THEORETICAL BACKGROUND

Solving the heat equation

For the solution of one dimensional heat equation (Equation (3.8)) we can use the technique of separation of variables, and assume a solution of the form [2].

\[ T(x, t) \propto \exp\left(i(kx - \omega t)\right). \quad (3.9) \]

**Q 4.** Show that a legitimate solution of the heat equation in the regime \( x \geq 0 \) is,

\[
T(x, t) = \sum_{\omega} A(\omega) \exp(-i\omega t) \exp\left(i - 1 \sqrt{\frac{\omega}{2D}} x\right) \\
= \sum_{\omega} A(\omega) \exp(-\sqrt{\frac{\omega}{2D}} x) \exp\left(i\sqrt{\frac{\omega}{2D}} x - \omega t\right). \quad (3.10)
\]

Fourier series of a square wave

In the present experiment we apply a periodic square pulse to a heater attached at one end of copper (C u) rod.

\[
\text{Figure 3.1: Sketch of a square pulse.}
\]

**Q 5.** Show that the Fourier series of a square wave, as shown in Figure 3.1, is given by,

\[
f(t) = A - \frac{2}{\pi} \sin(\omega_0 t) + \sin(3\omega_0 t) + \ldots \quad (3.11)
\]

where \( \omega_0 = \frac{2\pi}{T} \) is the fundamental frequency of the wave.

**Q 6.** What is the average value of the square wave?

**Q 7.** Observe the presence of only the odd harmonics in \( f(t) \). Can you verify the frequency components presented in a square wave using the concept of Fourier transform? Plot the Fourier transform of a simulated square pulse in Matlab.

**Q 8.** Explore the relationship between the Fourier transform and the Fourier series [2].
Applying boundary conditions

A particular solution to Equation (3.10) can be found if we apply the appropriate boundary conditions. In our case these conditions are determined by the square pulse heating waveform applied to the end of the rod, arbitrarily set at $x = 0$. Therefore,

$$T(0, t) = A \left( \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n+1} \sin((2n+1)\omega t) \right)$$

(3.12)

where $n = 1, 2, 3, \ldots$. Furthermore, after substituting $x = 0$ into Equation (3.10), we have,

$$T(0, t) = \sum_{\omega} A(\omega) \exp(-i\omega t).$$

(3.13)

**Q 9.** Identify the Fourier relationship between $T(0, t)$ and $A(\omega)$ in Equation (3.13)?

Equations (3.12) and (3.13) represent the same pulsing waveform and are necessarily equal. Hence, expressing (3.12) in complex form,

$$T(0, t) = A \left( \frac{1}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{e^{(2n+1)i\omega t} - e^{-(2n+1)i\omega t}}{2i} \right).$$

(3.14)

Comparing Equations (3.13) and (3.14), the only nonzero values that occur for $A(\omega)$ are,

$$A(0) = \frac{A}{2}$$

(3.15)

$$A((2n+1)\omega_0) = -\frac{i}{\pi(2n+1)}$$

(3.16)

$$A(-(2n+1)\omega_0) = \frac{i}{\pi(2n+1)}.$$  

(3.17)

Putting these conditions back into (3.10), we derive the following particular solution,

$$T(x, t) = A \left( \frac{1}{2} - \sum_{n=1}^{\infty} \frac{i}{(2n+1)\pi} \exp \left( -\sqrt{\frac{(2n+1)\omega_0}{2D}} x \right) \exp \left( i \sqrt{\frac{(2n+1)\omega_0}{2D}} x - \frac{(2n+1)\omega_0}{2D} t \right) \right)$$

$$+ \sum_{n=1}^{\infty} \frac{i}{(2n+1)\pi} \exp \left( -\sqrt{\frac{(2n+1)\omega_0}{2D}} x \right) \exp \left( -i \sqrt{\frac{(2n+1)\omega_0}{2D}} x + \frac{(2n+1)\omega_0}{2D} t \right) \right).$$

After some fairly simple algebraic reshuffling,

$$T(x, t) = A \left( \frac{1}{2} - \sum_{n=1}^{\infty} \frac{i}{(2n+1)\pi} \exp \left( -\sqrt{\frac{(2n+1)\omega_0}{2D}} x \right) \exp \left( i \sqrt{\frac{(2n+1)\omega_0}{2D}} x - \frac{(2n+1)\omega_0}{2D} t \right) \right)$$

(3.18)

$$+ \sum_{n=1}^{\infty} \frac{i}{(2n+1)\pi} \exp \left( -\sqrt{\frac{(2n+1)\omega_0}{2D}} x \right) \exp \left( -i \sqrt{\frac{(2n+1)\omega_0}{2D}} x + \frac{(2n+1)\omega_0}{2D} t \right) \right).$$

(3.19)
3.2. THEORETICAL BACKGROUND

Making the identifications, \((2n + 1)\omega_0 = \omega_n, - (2n + 1)\omega_0 = - \omega_n\), as well as,

\[
A(0) = \frac{A}{2}, \quad A(\omega_n) = \frac{-i}{\pi(2n+1)}, \quad A(-\omega_n) = \frac{i}{\pi(2n+1)} = (A(\omega_n))^* = A^*(\omega_n).
\]

the solution can be compactly written as,

\[
T(x, t) = \frac{A}{2} + \sum_{n=1}^{\infty} A(\omega_n) \exp(-i\omega_n t) \exp \left(i - 1 \sqrt{\frac{\omega_n}{2D}} x \right) \quad (3.20)
\]

\[
+ \sum_{n=1}^{\infty} A(-\omega_n) \exp(i\omega_n t) \exp \left(-i - 1 \sqrt{\frac{\omega_n}{2D}} x \right)
\]

\[
= \frac{A}{2} - \sum_{n=1}^{\infty} A(\omega_n) \exp(-i\omega_n t) \exp \left(i - 1 \sqrt{\frac{\omega_n}{2D}} x \right)
\]

\[
+ \sum_{n=1}^{\infty} A^*(\omega_n) \exp(i\omega_n t) \exp \left(-i - 1 \sqrt{\frac{\omega_n}{2D}} x \right) \quad (3.21)
\]

Since the second and third terms on R.H.S of Equation (3.21) are complex conjugates of each other, we can write,

\[
T(x, t) = \frac{A}{2} + 2 \sum_{n=1}^{\infty} A(\omega_n) \cos \left(\sqrt{\frac{\omega_n}{2D}} x - \omega_n t \right) \exp \left(-\sqrt{\frac{\omega_n}{2D}} x \right). \quad (3.22)
\]

Q 10. At a fixed \(x\), time harmonics of which order are present in the temperature oscillations?

Q 11. Define a damping coefficient \(\delta_n = \sqrt{\frac{2D}{\kappa}}\) for the \(n\)th harmonic. How does \(\delta_n\) vary with \(n\)?

The temperature oscillations shown in the solution (3.22) illustrate the superposition of damped oscillations.

Q 12. Calculate the ratio of the \(n\)th harmonic to the fundamental frequency.

It can be seen that the higher harmonics damp out very quickly because the damping coefficient increases with frequency. Therefore, at a sufficient distance from the origin, we can also approximate the temperature distribution through the first harmonic only.

Q 13. Write the temperature oscillation at a sufficiently long distance from the source of the heating, i.e., \(x \gg \delta_n\).
An everyday example

The surface of the Earth is heated by a diurnal temperature cycle that can be approximated by the sinusoidal variation

\[ T_0 + \Delta T \cos(\Omega t). \]  

(3.23)

where \( \Omega = 2\pi/24\ \text{h}^{-1}. \)

Q 14. How far into the Earth’s surface do the temperature oscillations penetrate?

3.3 The experiment

Figure 3.2 shows the experimental setup. A copper rod of length 1 m and diameter 30 mm with four thermocouples clamped to it, equidistantly arranged along the rod. The metal bar is heated with a square pulse, at a rate of 0.01 Hz using a cartridge heater which is inserted into the rod. The heater is a homemade 80 W iron. The heater is connected to a simple electric circuit shown in Figure 3.2. The relay is controlled using Labview, which sends a square pulse to the relay.

Q 15. How does a relay work and what is its purpose in the present experiment?

Scheme

K type thermocouples have been attached and the Labview code (Temperature oscillations.vi) has already been prepared. You have to
3.3. \textit{THE EXPERIMENT}

1. check the cold junction compensation (CJC) value.

2. apply a square pulse to the heater.

3. acquire the thermocouple data.

4. save the data to a file, and

5. plot the data and be able to investigate the following points. You are required to come up with suitable graphs to illustrate and respond to these points of inquiry.

Leave the system running for some time until the dynamic equilibrium state has been reached.

\textbf{Q 16.} How will you confirm that such a state is achieved?

\textbf{Q 17.} Plot all the thermocouple data in Matlab. What do you observe?

\textbf{Q 18.} Take the Fourier transform. What does the peak at zero frequency specify and how can it be removed?

\textbf{Q 19.} Determine the damping coefficient on each harmonic.

\textbf{Q 20.} Determine the ‘velocity’ of the ‘wave’.

\textbf{Q 21.} Determine the thermal diffusivity based on the phase velocity.

\textbf{Q 22.} Determine the thermal diffusivity based on the damping coefficient. Compare your diffusivity value with the published values [3].

\textbf{Q 23.} Compare your results from the preceding two equations and comment on the accuracy, and the relative accord (or discord) between them.

\textbf{Q 24.} Plot the Fourier transform of your data.

\textbf{Q 25.} Do you observe the higher frequency data to damp out more quickly? Back up your observations with quantitative results.

\textbf{Q 26.} What is the skin depth of the temperature oscillations?

\textbf{Comments on energy transfer}

You must have noticed that we have refrained from freely labelling these temperature oscillations as ‘thermal waves’. The first reason is that these oscillations
are solutions of the heat equation, not the wave equation which involves a second
time derivative. Heat conduction is a diffusive rather than a traveling wave. The
second reason which endorses this point of view is that these oscillations do not
transport energy [5].

Q 27. Using the temperature oscillations, Equation (3.22), determine the heat
transfer rate \( \mathcal{J} = -\kappa \nabla T \) through the Cu rod. Show that the so called thermal
'waves' do not carry energy and hence, they are fundamentally different from
sound waves or electromagnetic waves.

Q 28. Contrary to the theoretical suggestion, we know that as one end of the
Cu rod is heated the end does get hot. How does one resolve this paradox?

Q 29. Based on the data you have acquired, calculate the thermal conductivity
of Cu using Equation (3.1) and compare with your previously determined values.
Chapter 4

Principles and Applications of Superconducting Quantum Interference Devices

Asma Khalid, Rabiya Salman and Sabieh Anwar

Superconducting QUantum Interference Devices (SQUIDs) are sensitive devices that can detect small changes in the magnetic field. They take advantage of two important properties of superconductors, namely flux quantization and the Josephson effect. Furthermore, SQUIDs demonstrate quantum effects on the macroscopic scale, such as macroscopic wavefunctions, quantum interference and quantum mechanical tunneling. Mr. SQUID is a commercially available high temperature superconducting (HTS) DC SQUID magnetometer that will be used in our prototypical experiments.

KEYWORDS

Superconductivity · Josephson Junctions · Flux quantization · Meissner effect · Flux trapping · Phase order parameter · Superconducting ring · Critical current · Screening current

APPROXIMATE PERFORMANCE TIME: 2 weeks
4.1 Objectives

In this experiment, we will,

1. understand the phenomenon of superconductivity and macroscopic quantum behavior,
2. study the basic principles underlying SQUID applications,
3. observe zero resistance of superconductors and examine the superconducting phase transition,
4. observe the DC Josephson effect,
5. observe the periodically varying critical current in the resistive mode of the SQUID, and
6. learn about the detection of extremely small magnetic fields by converting them into voltages and using a feedback loop.
Bibliography


[5] http://cnx.org/content/m22750/1.3/, “Theory of A Superconducting Quantum Interference Device (SQUID)”.


4.2 Theoretical Introduction

Superconductors

A superconductor is an element, intermetallic alloy or a compound (may be organic) that loses its electrical resistance below a transition temperature $T_C$. Once
setup, the super-currents in these materials have been observed to flow without measurable decrease for several years [1].

Superconductors can be categorized into type I and type II superconductors. The type I superconductors mainly comprising of metals and metalloids that require incredibly low temperatures to superconduct. On the other hand, type II superconductors comprise of metallic compounds, alloys, and cuprates. They achieve higher $T_C$'s than type I superconductors and are more promising for practical applications. An important difference between the two types is that in that in type II, the transition from a normal to a superconducting state is gradual and involves a region of "mixed state" behavior.

In all superconductors, the current is carried not by single electrons but by pairs of electrons with opposite spins called Cooper pairs. These are quantum mechanical entities. For $T < T_C$, the binding energy of the Cooper pair is large as compared to the thermal energy scattering. As a result, Cooper pairs propagate through the material and current flows without any resistance. The typical transition from normal to superconducting behavior is shown in Figure 4.1.

![Resistance-Temperature Curve](image)

Figure 4.1: Resistance-temperature curve for Superconductors, $\delta R/\delta T|_{T=T_c} \to \infty$.

**Characteristics of Superconductors**

**Infinite conductivity**

Superconductors exhibit the remarkable property of infinite conductivity. Within experimental error, their resistivities are found to be lower than $10^{-26} \ \Omega \cdot m$ at $T < T_C$. For comparison the resistivity of copper, one of the finest conductors is at the most $10^{-8} \ \Omega \cdot m$ at room temperature.
4.2. THEORETICAL INTRODUCTION

The well known relationship of current density is,

\[ j = \sigma E. \]

Since the conductivity is infinite, to have finite values of \( j \) the electric field \( E \)
should be zero inside a superconductor. But according to Faraday's law,

\[
-\frac{\partial B}{\partial t} = \nabla \times E
\]

\[ = 0 \]

\[ \Rightarrow B = \text{constant.} \quad (4.1) \]

Hence inside infinite conductivity materials including superconductors, the magnetic field \( B \) is constant in time. The property that differentiates a superconductor from a normal perfect conductor is that the magnetic field \( B \) is not only independent of time but is also zero inside a superconductor. This phenomenon is discussed in more detail in Section 2.2.4.

Cooper pairs

Cooper pair is the name given to the pair of electrons that are bound together at low temperatures. The American physicist Leon Cooper showed that an arbitrarily small attraction between electrons in a metal can cause a paired state of electrons to have a lower energy than the Fermi energy, which implies that the pair is bound. In superconductors, this attraction is due to the electron—phonon interaction. According to the BCS theory of superconductivity, these Cooper pairs are responsible for superconductivity.

The BCS theory is the first microscopic theory of superconductivity, proposed by Bardeen, Cooper, and Schrieffer in 1957, almost 46 years after the discovery of superconductivity in 1911. It describes superconductivity as a microscopic effect caused by the "condensation" of pairs of electrons into the bosonic Cooper state.

The Meissner Effect

The Meissner effect is the exclusion of magnetic flux from the superconductor. This is due to the electric currents known as the screening currents flowing on the surface of the superconductor. The screening currents flow in such a direction so as to generate a field equal and opposite to the applied field. This results in \( B = 0 \) inside the superconductor. Superconductors expel the field even if they are cooled into the superconducting state in the presence of an applied field as shown...
in Figure 4.2. This behavior is in contrast with a normal, infinitely conducting sample.

![Diagram](image-url)

**Figure 4.2:** (a) A normal conductor cooled in the presence of magnetic field tends to sustain that field after transition to zero resistance. (b) Circular screening currents expels the magnetic field as the superconductor is cooled below its transition temperature.

Considering the superconductor from the point of view of a magnetic material in which screening currents produce an internal magnetic field \( \mu_0 M \) expelling the applied field \( \mu_0 H \), we require for the Meissner effect

\[
B = \mu_0 (H + M) = 0
\]

\[\Rightarrow M = -H.\]

Hence the equation relating magnetization and applied magnetic field strength,

\[M = \chi H\]

shows that the superconductor behaves as though it has a magnetic susceptibility \( \chi = -1 \), a perfect diamagnet.

**Meissner effect in type II superconductors** A type II superconductor allows some of the external magnetic field to penetrate into its surface which creates some rather novel phenomena like superconducting "stripes" and "flux-lattice vortices".

**London equation and description of the Meissner effect**

To understand the London equation, we first need to understand the concept of electromagnetic momentum. Consider a stationary particle of charge \( q \) and mass
4.2. **THEORETICAL INTRODUCTION**

$m$ at a distance $r$ from the axis of a long solenoid. For simplicity we consider the solenoid to be superconducting with its ends connected together to form a complete circuit as shown in Figure 4.3.

![Figure 4.3: A charged particle accelerated by a decaying magnetic field.](image)

Initially the solenoid is at $T < T_C$ and is carrying a supercurrent. The charge $q$ is in a field-free region as all the field is concentrated inside the solenoid. Now if the solenoid is heated to $T > T_C$, the current and hence the magnetic field decays. By virtue of Faraday's law, a changing magnetic flux induces an emf around the loop C accelerating the charge particle, giving it a momentum $mv$. Where does this momentum come from? After all, we have not applied any force on the particle to boost its momentum from 0 to $mv$. Are we violating conservation of momentum here?

This paradox can be resolved by arguing that the particle possessed the momentum throughout! For the law of conservation of momentum to hold, we write the momentum of the particle as

$$\mathbf{p} = n\mathbf{v} + q\mathbf{A}$$  \hspace{1cm} (4.2)

where $\mathbf{p}$ is the so-called canonical momentum, $q$ is the charge and $\mathbf{A}$ is the magnetic vector potential ($\nabla \times \mathbf{A} = \mathbf{B}$) [2]. The momentum $q\mathbf{A}$ possessed by a charged particle at rest is called the electromagnetic momentum. In the initial state, all the momentum of the charge particle is electromagnetic and in the final state, it is wholly kinetic.

Considering the current decay period to be very short, we can express the kinetic
momentum as an impulse of force. Thus we can write the kinetic momentum as,

\[ \mathbf{m}\mathbf{v} = \int_C \mathbf{F}_{\text{elec}} dt \]  
\[ = \int_C q\mathbf{E} dt, \]  

where \( \mathbf{F}_{\text{elec}} \) is the electric force on charge \( q \). The expression for the electric field at the position of the particle can be written using the Faraday’s law,

\[ \int_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\phi_B}{dt}, \]

where \( \phi_B \) is the magnetic flux through the circular loop \( C \).

The magnetic field \( \mathbf{B} \) at the position of particle is zero owing to the long size of the solenoid. The above equation can also be expressed in terms of \( \mathbf{A} \) (which is non-zero at the position of the particle) using the expression for magnetic flux \( \phi_B \), and Stoke’s theorem that relates a line integral to an area integral. Hence the equation is transformed into,

\[ \int_C \mathbf{E} \cdot d\mathbf{l} = -\int_C \frac{\partial \mathbf{A}}{\partial t} d\mathbf{l} \]
\[ = -\frac{d\mathbf{A}}{dt}. \]

Equation 4.4 can then be written as,

\[ \mathbf{m}\mathbf{v} = -q \int \frac{d\mathbf{A}}{dt} dt \]
\[ = -q \Delta \mathbf{A}_{t_f}^{t_i} \]
\[ = -q(\mathbf{A}(t_f) - \mathbf{A}(t_i)) \]
\[ = q\mathbf{A}(t_i). \]

The above equation shows that even though the particle is kinetically accelerated, its canonical momentum defined by Equation 4.2 is indeed conserved.

Comparing the initial \((T < T_C)\) and final momenta \((T > T_C)\),

\[ \mathbf{p}_i = \mathbf{p}_f \]

\[ \mathbf{m}\mathbf{v}(t_i) + q\mathbf{A}(t_i) = \mathbf{m}\mathbf{v}(t_f) + q\mathbf{A}(t_f) \]
\[ q\mathbf{A}(t_i) = \mathbf{m}\mathbf{v}(t_f). \]

Since \( \mathbf{v}(t_f) = \mathbf{A}(t_f) = 0 \), we get a general expression for the drift velocity of electron as,

\[ \mathbf{v}_d = -\frac{e\mathbf{A}}{m_e} \]  
\[ (4.5) \]
and the current density for the electrons turns out to become,

\[ \mathbf{j} = n \mathbf{v} e \]
\[ = -\frac{n \mathbf{e}^2}{m_e} \mathbf{A}. \]

(4.6)  \hspace{1cm} (4.7)

Taking the curl of the above equation and replacing the electron current density \( n \) by the superconducting current density \( n_s \), we arrive at the London equation for superconductors,

\[ \nabla \times \mathbf{j} = -\frac{n_s \mathbf{e}^2}{m_e} \mathbf{B}. \]

(4.8)

We now set to show that Equation 4.8 implies the screening of the magnetic field by the superconducting electrons. We take the curl of the Maxwell equation

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{j}. \]

(4.9)

Using the vector identity,

\[ \nabla \times \nabla \times \mathbf{B} = \nabla (\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} \]

(4.10)

and Equations 4.9 and 4.10, we arrive at,

\[ -\nabla^2 \mathbf{B} = \mu_0 \mathbf{j} \]
\[ = -\frac{\mu_0 n_s e^2}{m_e} \mathbf{B}. \]

(4.11)  \hspace{1cm} (4.12)

Q 1. Derive Equation 4.12 using Equations 4.9 and 4.10.

The above equation 4.12 can also be expressed as,

\[ \nabla^2 \mathbf{B} = \frac{\mu_0 n_s e^2}{m_e} \mathbf{B} \]
\[ = \frac{1}{\lambda_L^2} \mathbf{B}. \]

(4.13)

where

\[ \lambda_L = \frac{m_e}{\mu_0 n_s e^2} \]

is a characteristic length called the London penetration depth.

The above solution shows that \( \mathbf{B} \) decays exponentially as we go into the interior of the superconducting region. Let the external magnetic field be applied along the \( z \)-axis parallel to the surface of a thin slab of superconductor (thickness \( x \approx 0 \)), we have for this one dimensional setup,  

\[ \frac{d^2 \mathbf{B}}{dx^2} - \frac{\mathbf{B}}{\lambda^2} = 0. \]

(4.14)

Applying the boundary conditions \( \mathbf{B} = B_{ext} \) at \( x=0 \) and \( \mathbf{B}=0 \) at \( x=\infty \), we can get the solution of above differential equation as

\[ \mathbf{B}(x) = B_{ext} \exp \left( -\frac{x}{\lambda} \right). \]

(4.15)
This is the London equation indicating that the magnetic field exponentially decays to zero inside a superconductor. The magnetic flux thus penetrates the sample only for a small distance from the surface and becomes zero at \( x \gg \lambda \). The length scale of the penetration is determined by \( \lambda \).

**Q 2.** Why do we take divergence of \( \mathbf{B} \), \( \nabla \cdot \mathbf{B} \) as zero?

**Q 3.** Solve the second order differential equation 4.14 and apply the appropriate boundary conditions to derive the solution 4.15.

**Q 4.** Formulate the London equation in terms of the current density \( \mathbf{j} \) similar to Equation 4.13. What do you conclude from this equation? HINT: Use Equation 4.8 along with the Maxwell \( \nabla \times \mathbf{B} = \mu_0 \mathbf{j} \) and the continuity equation \( \nabla \cdot \mathbf{j} = 0 \).

**Q 5.** What is the physical significance of the continuity equation, \( \nabla \cdot \mathbf{j} = 0 \)?

**Macroscopic quantum phenomena in superconductors**

Cooper pairs in a superconductor share a common wavefunction \( \psi(r) \) and the behavior of the superconducting electrons is completely specified by this function. This is in complete contrast to the situation in a normal metal where the behavior can only be determined by specifying all of the occupied single-particle states. This coherence in wavefunction associated with macroscopic occupation of the same state by Cooper pairs causes a superconductor to directly manifest quantum mechanics at a large scale! How pleasing!

The macroscopic wavefunction is specified by the order parameter.

\[
\psi(r) = \psi_0 \exp(i \mathbf{k} \cdot \mathbf{r}). \quad (4.16)
\]

Here \( \mathbf{k} \) is the wave vector and \( \mathbf{r} \) is the position vector. The Cooper pair density is,

\[
|\psi(r)|^2 = \psi_0^2 = n_s/2.
\]

This means that if the electronic density is \( n_s \), then the Cooper pair density is \( n_s/2 \). Several interesting quantum phenomena can now be motivated from the order parameter.

**Q 6.** A normal metal has \( N \) non-interacting conduction electrons. These electrons have wavefunctions \( \psi_1(r_1), \psi_2(r_2), \ldots, \psi_N(r_N) \). What is the combined wavefunction of all electrons?
4.2. THEORETICAL INTRODUCTION

Q 7. What is the physical significance of the wave vector \( \mathbf{k} \)? How is it related to the momentum?

**Supercurrent density derivable from the wavefunction**

The canonical momentum for the Cooper pairs is

\[
\mathbf{p} = 2m_e \mathbf{v} - 2e \mathbf{A} \tag{4.17}
\]

From equations 4.6 and 4.17 we obtain,

\[
\mathbf{j}_e(r) = -\frac{n_e}{2} \left( \frac{(\mathbf{p} + 2e \mathbf{A})}{2m_e} \right) = \frac{-e}{m_e} |\psi(r)|^2 (\mathbf{p} + 2e \mathbf{A}) = \frac{-e}{m_e} \psi^*(\mathbf{p} + 2e \mathbf{A}) \psi.
\]

Substituting the momentum operator by its differential operator expression \(-i\hbar \nabla\) we get,

\[
\mathbf{j}_e(r) = \frac{e}{m_e} \left[ i\hbar \psi^*(r) \nabla \psi(r) - 2e \mathbf{A} \psi^*(r) \psi(r) \right].
\]

This expression is, in general, complex. To get a real expression for the current density, we add the complex conjugate of first term on the right hand side of above equation and take its average,

\[
\mathbf{j}_e(r) = \frac{e}{m_e} \left[ \frac{(i\hbar \psi^*(r) \nabla \psi(r)) + (i\hbar \psi^*(r) \nabla \psi(r))^\dagger}{2} - 2e \mathbf{A} \psi^*(r) \psi(r) \right] = \frac{e\hbar}{2m_e} [(\psi^*(r) \nabla \psi(r) - \psi(r) \nabla \psi^*(r)) - \frac{2e^2}{m_e} \mathbf{A} \psi^*(r) \psi(r)] \tag{4.18}
\]

which is an expression for the superconducting current density based on the order parameter.

In its most general form we write the order parameter as

\[
\psi(r) = |\psi(r)| \exp(i\alpha(r))
\]

where \( \alpha(r) = \mathbf{k} \cdot \mathbf{r} \) is a phase.

Q 8. Defining the gradient operator \( \nabla \) in spherical polar coordinate system \((r, \theta, \phi)\) and using the above expression for \(\psi(r)\) in Equation 4.16, derive the following alternative expression for \(\mathbf{j}_e(r)\),

\[
\mathbf{j}_e(r) = -\frac{e}{m_e} |\psi(r)|^2 \left( \hbar \nabla \alpha + 2e \mathbf{A} \right). \tag{4.19}
\]

Q 9. Derive the London Equation 4.8 by taking curl of Equation 4.19 assuming that the Cooper pair density is independent of position, i.e., \( \psi(r)^2 = n_e/2 \).
Flux quantization

![Superconducting ring](image)

Figure 4.4: Closed superconducting ring containing a closed path $C$ far from the surface.

Let's consider a superconductor fashioned in the form of a ring as shown in Figure 4.4. The superconductor is in its Meissner state and allows a circular path along $C$, far from the surface, the current density $\mathbf{j} = 0$. Equation 4.19 thus becomes,

$$\hbar \nabla \alpha = -2e \mathbf{A}.$$  

We integrate both sides of this equation around the closed curve $C$, apply Stoke's theorem and use the definition of the magnetic flux

$$\hbar \oint_{C} \nabla \alpha \cdot d\mathbf{l} = \hbar \Delta \alpha$$

$$= -2e \oint_{C} \mathbf{A} \cdot d\mathbf{l}$$

$$= -2e \int_{s} \left( \nabla \times \mathbf{A} \right) \cdot d\mathbf{s}$$

$$= -2e \int_{s} \mathbf{B} \cdot d\mathbf{s}$$

$$\implies \hbar \Delta \alpha = -2e \phi$$

$$\phi = -\frac{\hbar}{2e} \Delta \alpha. \quad (4.20)$$

Here $\Delta \alpha$ is the change in phase of the order parameter as one makes a complete round trip along $C$. Since the order parameter $\psi(r)$ is a legitimate wavefunction, it must be single-valued and the phase change $\Delta \alpha$ around the closed loop must be $\pm 2\pi n$, where $n$ is an integer (positive or zero). Thus,

$$\phi = \pm \frac{2\pi n \hbar}{2e}$$

$$= \pm \frac{n \hbar}{2e}$$

$$= \pm n \phi_0,$$
which shows that the magnetic flux through any closed area within a superconductor, on whose perimeter \( j = 0 \), is quantized in units of the flux quantum, \( \phi_0 = \hbar/2e = 2.07 \times 10^{-15} \text{Tm}^2 \). The flux in quantized just like charge or spin quantization. The smallest unit \( \phi_0 \) is, quite aptly called the “fluxon”.

The DC Josephson effect

The Josephson effect is a manifestation of long-range quantum coherence of superconductors. Josephson was a Ph.D. student in Cambridge when he discovered this phenomenon. It occurs when two superconducting regions are weakly coupled. A Josephson junction (JJ) is formed by placing an insulating gap between two superconductors. If the gap is thin enough, electron pairs can tunnel from one superconductor across the gap to the other superconductor. By quantum tunneling, a resistanceless current can flow across the insulator. This is called the DC Josephson effect.

Figure 4.5: Schematic diagram of two superconducting regions separated by a thin gap.

To understand this effect, we consider two isolated samples of a superconductor with spatially constant order parameters, \( \psi_1 \) and \( \psi_2 \) as shown in Figure 4.5. In general, \( \psi_1 \neq \psi_2 \). Let the order parameter in the left region be \( |\psi_1\rangle \exp(i\alpha_1) \) and that on the right be \( |\psi_2\rangle \exp(i\alpha_2) \). In the absence of interaction between the two samples the phases \( \alpha_1 \) and \( \alpha_2 \) will in general be different. Strongly coupling the two samples by bringing them into contact over a large area causes the phases to equalize \( \alpha_1 = \alpha_2 \), so that all Cooper pairs will be in the same state; this equality is then very difficult to disturb. But if there is a weak coupling, the phases \( \alpha_1 \) and \( \alpha_2 \) will not equalize between the two regions. It is possible to maintain a phase difference between the two regions by passing a small current through the JJ. This is the DC Josephson effect.

Below \( T_C \) it is possible for the Cooper pairs to tunnel through the barrier, a net flow can take place even in the absence of an applied potential difference. This corresponds to a dissipation-less supercurrent whose density is calculated below. Because of the tunneling of pairs, the superconducting order parameter extends throughout the barrier. Inside the barrier, we regard it as being the sum of the
Figure 4.6: Contribution to the superconducting order parameter within the oxide barrier associated with the tunneling of Cooper pairs through the barrier.

contributions shown in Figure 4.6. We thus write the order parameter inside the barrier as,

$$\psi = |\psi_0| \exp(i\alpha_1 - k(x + d/2)) + \exp(i\alpha_2 + k(x - d/2))$$

$$= (n_s/2)^{1/2}\exp(i\alpha_1 - k(x + d/2)) + \exp(i\alpha_2 + k(x - d/2)))(4.21)$$

where the barrier extends from $x = -d/2$ to $x = d/2$ and $k^{-1}$ is the characteristic length for decay of the order parameter within the barrier. Furthermore, we assume $|\psi_1|^2 = |\psi_2|^2 = n_s/2$.

Q 10. Calculate the pair current density through the barrier by using the order parameter of Equation 4.21 and $A = 0$. The phases $\alpha_1$ and $\alpha_2$ are assumed to be spatially constant within the superconducting region.

The expression you achieve should be similar to,

$$j = \frac{\hbar n_s}{2}\exp(-kd)[\exp(i(\alpha_1 - \alpha_2) + \exp(i(\alpha_2 - \alpha_1)]$$

$$= j_0 \sin(\alpha_1 - \alpha_2)$$

(4.22)

where

$$j_0 = \frac{\hbar ke^{-kd}}{m_e} n_s. \quad (4.23)$$

The expression shows that the difference in the phase factor on either side, leads to a tunneling current, even when there is no applied potential difference. The tunneling current is a supercurrent. It is possible to directly observe the DC Josephson effect, and in the process, quantum mechanical tunneling in the present experiment. SQUID provides a highly inspirational way of practically observing tunneling!
4.2. THEORETICAL INTRODUCTION

The SQUID as a magnetometer

A superconducting quantum interference device (SQUID) uses the properties of flux quantization and the DC Josephson effect to detect very small magnetic fields. They are sensitive enough to measure fields down to the range of $10^{-15}$ T. For comparison, a typical refrigerator magnet produces 0.01 T, and some processes in animals produce very small magnetic fields between $10^{-9}$ T and $10^{-6}$ T.

The central element of a SQUID is a ring of superconducting material with one or more JJs. An example is shown in Figure 4.7. The critical current, $I_c$, of the junctions is much less than the critical current of the main ring. This produces a very low current density making the momentum of the electron-pairs small.

![Figure 4.7: A DC SQUID in the presence of an applied magnetic field.](image)

To understand the working of SQUID, let's consider the scenario when we bias it with a current well below its critical current value. Then, if we apply a tiny magnetic field to the SQUID, the applied magnetic field tends to change the superconducting wave function $\psi(r)$. But the superconducting wavefunction doesn't want to change. As discussed in Section 4.2, it must maintain an integral number of wavefunction cycles around the loop. So the superconducting loop does what you would expect; it opposes the applied magnetic field by generating a screening current $I_s$, that flows around the loop as shown in Figure 4.7. The screening current creates a magnetic field equal but opposite to the applied field, effectively canceling out the net flux in the ring.

In this way, the applied magnetic field has lowered the effective critical current
of the SQUID. In other words, it has reduced the amount of bias current we can pass through the SQUID without generating a resistive voltage, since the screening current has superimposed itself on top of the bias current. The situation is depicted in Figure 4.7, where $I_s$ flows parallel to $I_{bias}/2$ in one of the arms, increasing the total current, making it more likely to achieve the superconducting to normal transition.

Now, as we increase the applied magnetic flux, the screening current increases. But when the applied magnetic flux reaches half a flux quantum, something interesting happens: the superconducting junctions momentarily become resistive. If we try to observe this transition in terms of screening current, we can conclude that in order to have lower energy at $\phi \approx \phi_0/2$, it is little easier for the SQUID to keep 0.49 flux quanta in rather than keeping 0.51 flux quanta out. Of course, the screening current will have to change direction at this point. The variation of $I_s$ with $\phi$ is shown in Figure 4.8.

![Figure 4.8: Relationship between screening current and applied magnetic flux. (This figure is taken from [3] page no. 37.)](image)

Figure 4.8 clearly shows that the screening current changes sign when the applied flux reaches half of a flux quantum. Then, as the applied flux goes from half a flux quantum toward one flux quantum, the screening current decreases. When the applied flux reaches exactly one flux quantum, the screening current goes to zero. At this instant, the magnetic flux inside the loop and the magnetic flux applied to the loop are equal, so there is no need for a screening current. If we increase the applied magnetic flux a little more, a small screening current starts to flow in the positive direction, and the cycle begins again. The screening current is periodic in the applied flux, with a period equal to one flux quantum, $\phi_0$.

Hence we conclude that,

- the screening current of a SQUID is periodic in the applied flux, and
• the critical current of a SQUID depends on the screening current.

Thus it makes sense that the SQUID critical current is also periodic in the applied magnetic flux. The critical current goes through maxima when the applied magnetic flux is an integral multiple of the flux quantum \( I_c = 0 \) and it goes through minima when the applied magnetic flux is an integral multiple of one plus half flux quantum.

The critical current is usually obtained by measuring the voltage drop across the junction as a function of the total current through the device. To make a magnetometer, the SQUID is operated at a biasing current slightly greater than \( I_c \), so the SQUID always operates in the resistive mode. Under these conditions, there is a periodic relationship between the voltage across the SQUID and the applied magnetic flux with a period of one flux quantum \( \phi_0 \). This is shown in Figure 4.9.

![Figure 4.9: Voltage-Flux characteristics of DC SQUID showing the periodic dependence of the SQUID voltage on applied flux for a fixed bias current.](image)

**Q 11.** The periodically varying critical current as a function of applied magnetic flux can be expressed mathematically as

\[
I_c(\phi) \propto I_c(0) \left| \cos \left( \frac{\pi \phi}{\phi_0} \right) \right|.
\]  

(4.24)

Derive Equation 4.24 from [6], page no. 308-310. How will the equation’s graphical depiction differ from Figure 4.9?

The sinusoidally varying critical current is, in fact, a direct demonstration of quantum interference. The variation in the current can be compared to the variation in the intensity of light detected on the screen of a Young’s double-slit experiment. In the latter experiment, a phase difference is achieved by varying the
optical path length, as we remember from our basic physics course. In the case of the SQUID, the phase difference is maintained by the currents flowing through the two arms of the SQUID. Refer to [6], page 310 for details. The phenomena of interference and flux quantization are exploited to create the world’s most sensitive magnetic field detectors [3, 5]. These ideas will be directly verified in the present experiment.

Remember, when using the SQUID as a highly sensitive superconducting detector for magnetic flux, we bias it with a current slightly higher than the critical current \( I_C \) so that a voltage drop occurs across the JJ's with the SQUID operative in the resistive mode.

4.3 Overview of the Mr. SQUID apparatus

Our experiment uses “Mr. SQUID”, a commercial DC Superconducting Quantum Interference Device (SQUID) magnetometer, incorporating the following components,

- a high-temperature superconductor (HTS) thin-film SQUID chip,
- two feedback coils to modulate the SQUID and to couple an external signal to the SQUID
- a cryogenic probe with a removable magnetic shield
- an electronic control box containing all the circuits needed to operate the SQUID, and
- a cable to connect the probe to the electronics box.

A schematic diagram of Mr. SQUID with the above mentioned components is shown in Figure 9.2.

At the heart of Mr. SQUID is a small integrated circuit chip made of yttrium barium copper oxide (YBCO) that is fashioned into a ring containing two Josephson junctions. For a full description and working of this device, refer to the relevant sections of the Mr. SQUID manual [3].
Figure 4.10: A simplified schematic diagram of the Mr. SQUID experiment. Current flowing into the superconducting ring is monitored by an ammeter ‘A’ in series and voltage developed across the JJs can be detected by a voltmeter ‘V’ connected in parallel. Current in the external and internal modulation coils links flux to the ring.

Figure 4.11: Photograph of the Mr. SQUID probe along with its electronic box.

**Brief description to the Electronic Box**

**POWER switch**  The power switch of the SQUID’s control box is normally in the OFF (down) position.

**V – I, V–φ mode switch**  The mode switch is normally in the V – I (up) position. This mode allows us to observe the voltage-current characteristics of the SQUID, i.e., we apply a biasing current I to the SQUID and observe the voltage drop V across the parallel JJs. In this mode we can directly observe...
the DC josephson effect. The $V - \phi$ (down) position switch allows us to observe the voltage-flux characteristics of the SQUID, where $\phi$ is the externally applied magnetic flux and $V$ is the periodically changing voltage across the SQUID as the flux is varied. The period of the voltage modulation is determined by the quantum of the magnetic flux, the fluxon.

**PROBE**  Nine-pin DB-9 socket for connections to the Mr. SQUID probe.

**BIAS OFFSET knob**  This knob applies a fixed DC current through the SQUID. In the 12 o'clock position, this current is approximately zero. Turning the knob either clockwise or anticlockwise supplies a fixed current through the SQUID in either of the two directions.

**EXT INPUT**  A BNC connector used to couple an external voltage signal to the "external" feedback coil on the Mr. SQUID chip. A MODE switch inside the Mr. SQUID electronics box (accessible by removing the top cover) selects whether this signal is routed directly through a 100 mA fuse at location F1 inside the box to the "external" feedback coil on the Mr. SQUID chip (switch position DIR) or is converted to true differential input using a buffer amplifier and then routed to the feedback coil (switch position BUF). The current output from the buffer amplifier (i.e., the current applied to the external coil) is 100 $\mu$A/V. The buffered configuration is the default factory setting.

**SWEEP OUTPUT knob**  This control sets the amplitude of the triangle wave test signal in either the $V - I$ or the $V - \phi$ mode. The control sweeps the current back and forth between the two extreme values. These values are set by rotating the position of this knob while the center position of the current is determined by the BIAS OFFSET knob. This arrangement is shown in Figure 4.12. In the $V - I$

![Image of BIAS and SWEEP outputs](image.png)

Figure 4.12: Function of the BIAS and SWEEP outputs of the SQUID electronic box.

In the $V - I$ mode, the triangular current wave is applied to the bias terminals of the SQUID.
and in the $V-\phi$ mode, the triangular wave is applied to the internal modulation coil.

**FLUX OFFSET knob**  
This knob applies a fixed DC current to the internal modulation coil and thus acts as a source of applied magnetic field for the SQUID. In the 12 o'clock position this current is approximately zero. This function controls the amount of magnetic flux through the central annular region in the SQUID loop.

**CURRENT Output**  
A BNC female connector provides the output current of the Mr. SQUID box. The current is converted to an output voltage by dropping across a 10kΩ resistor. Hence to determine the current flowing through the JJs, we divide the measured voltage by 10kΩ.

- In the $V-I$ mode this output represents the total current through the SQUID (sum of the fixed bias current and triangular wave provided by BIAS OFFSET and SWEEP OUTPUT respectively).

- In the $V-\phi$ mode this output represents the current through the modulation coil (sum of fixed modulation current through the FLUX OFFSET control and the triangular wave).

**VOLTAGE Output**  
Another BNC female connector provides the voltage across the SQUID (in both the $V-I$ and $V-\phi$ modes). This voltage is amplified by a factor of 10,000. So the actual voltage across the SQUID is the measured voltage divided by 10,000.

## 4.4 Experiments

### The DC Josephson effect and $V-I$ characteristics of the SQUID

**Objective**

In this experiment we will observe the resistanceless current in the SQUID and also measure the critical current.

**Apparatus**

1. Mr. SQUID probe
2. Electronic control box
3. DB-9 M/M cable with 9-pin male connector
4. Liquid nitrogen
5. Magnetic shield

Procedure

1. Configure the oscilloscope for the dual X-Y mode and set the vertical sensitivity to 0.5 V/div and the horizontal sensitivity to 0.1 V/div. With the GND on the scope, use the horizontal and vertical positioning knobs to set the CRO dot to the middle of the screen.

2. Using BNC cables connect the VOLTAGE output of the SQUID's control box to the Y-channel of the CRO and the CURRENT output to the X-channel and set the source switch to CH 1.

3. Confirm that the POWER switch of the electronic box is OFF and the mode switch is in the V – I position, FLUX OFFSET and BIAS OFFSET are in the 12 o’clock position and the SWEEP OUTPUT knob is fully counterclockwise.

4. Plug the 5-pin power cable into the POWER connector at the rear panel of the electronic box.

5. Plug one end of the 9-pin DB-9 M/M cable into the PROBE connector and the other end into the Mr. SQUID probe.

6. Now turn the POWER switch ON and switch the CRO input coupling to DC mode. Use the BIAS OFFSET knob to position the dot in the middle of the CRO screen. In case you are unable to achieve this, check the connections and see Section 6 of the Mr. SQUID (page no. 49-55) manual.

7. Fill the dewar about 3/4-full with LN2. Fix the shielded probe into its styrofoam black cover and slowly lower the sensor end into the LN2 dewar.

8. The critical temperature for the YBCO superconductor in SQUID is \( T_c \approx 90\text{K} \) and it will take some minutes for the SQUID sensor to reach the boiling point of LN2, 77K.

9. It is important to cool the SQUID while minimizing the presence of magnetic fields in the surroundings. This will reduce the effect of flux trapping.
10. Minimize the BIAS OFFSET control and increase the SWEEP OUTPUT.
The \( V - I \) curve will appear on the CRO screen. Use the BIAS OFFSET
to symmetrize the trace and FLUX OFFSET to maximize the supercurrent
(zero voltage drop) region. A good \( V - I \) curve should look like that shown
in Figure 4.13.

![Image of V-I characteristics of SQUID with CH I at 0.1 V/div and CH II at 0.5 V/div.]

**Figure 4.13: V – I characteristics of SQUID with CH I at 0.1 V/div and CH II at 0.5 V/div.**

**Determining the critical current**

The \( V - I \) curve has a flat region at the center where the current is flowing with
zero voltage drop. This region represents the supercurrent or zero-resistance cur-
rent flowing through the JJ's, exhibiting DC Josephson effect. The current exists
even though there is no voltage drop. You must have guessed that the current
flows because of a nonzero phase difference \( \Delta \phi \) between the two superconduc-
ting regions across the JJ. But the superconductor ceases to be resistanceless as
soon as the current exceeds the critical current, \( I_C \). This critical current can be
determined through the procedure given below.

![Image of method to determine the critical current 2I_c.]

**Figure 4.14: Method to determine the critical current 2I_c.**

1. Measure the voltage at the “knee” in the \( V - I \) curve from the CRO screen
after maximizing the zero voltage current by using the FLUX OFFSET
control.
2. Divide that voltage by 10,000 to convert the voltage to current.

3. The \( V - I \) curve is typically rounded owing to the thermal noise. To measure the current more accurately, increase the vertical and horizontal sensitivities at the CRO channels.

4. As shown in Figure 4.14, extrapolate the straight part of the \( V - I \) curve in the resistive region down to the horizontal axis. The point of intersection with the horizontal axis corresponds roughly to the critical current in the absence of thermal noise.

5. However this current is through both the junctions in the SQUID, so the critical current through one Josephson junction is half this value.

**Calculation of the normal state resistance \( R_N \)**

The \( V - I \) mode also helps us measure another parameter of the JJ known as the normal state resistance. It can be measured by calculating the slope of the \( V - I \) curve in the resistive region of the SQUID as described below.

1. Increase the horizontal and vertical sensitivities on the CRO channels

2. Increase the SWEEP OUTPUT control almost to the point where the horizontal and vertical outputs saturate, i.e., the ends of the \( V - I \) curve get clipped and appear as flat lines as shown in Figure 4.15.

![Figure 4.15: Saturation in the \( V - I \) curve as the SWEEP OUTPUT is increased.](image)

3. The slope of an imaginary line drawn between the endpoints of the \( V - I \) curve passing through the origin, corresponds to the normal resistance of the SQUID. Refer to Figure 4.16.

4. Since the SQUID ring contains two JJs and hence two resistances in parallel, so the normal state resistance for a single JJ would be twice the slope of the dashed line drawn in Figure 4.16.
4.4. **EXPERIMENTS**

![Diagram](image)

Figure 4.16: Method to determine the normal resistance of the JJ. $R_N$ is twice the slope of the dashed line.

### Flux quantization and $V$-$\phi$ characteristics of the SQUID

**Objective**

This experiment investigates the quantization of flux through SQUID ring. We will observe one of the most remarkable properties of the DC SQUID: the development of a periodic voltage across it in response to an applied magnetic flux, when biased slightly above the critical current.

**Q 12.** Equation 4.24 shows the sinusoidal variation of the critical current with flux. What is your prediction about the voltage developed across the SQUID as a function of the flux?

**Apparatus**

1. Mr. SQUID probe
2. Electronic control box
3. DB-9 M/M cable with 9-pin connector
4. Liquid nitrogen
5. Magnetic shield

**Procedure**

1. In the $V$ -- $I$ mode, rotate the SWEEP OUTPUT control completely counterclockwise (so as to see a dot on the CRO). Now rotate the BIAS OUTPUT knob, the dot will trace the usual $V$ -- $I$. Bias the SQUID just slightly
above the critical current by adjusting the dot slightly above the knee of the $V-I$ curve. This is the most sensitive point on the $V-I$ curve.

2. Manually modulate the SQUID with the FLUX OFFSET control so that the point on the CRO screen will move up and down in response to the changing flux as an integral multiple of fluxons threading the superconducting ring. This periodic motion arises because the screening current in the SQUID body depends on the applied magnetic flux in a periodic manner where the period is determined by the magnetic flux quantum ($\Phi_0$).

3. Turn the MODE switch to $V-\phi$, the down position. We will now attempt to automate the above step.

4. Rotate the SWEEP OUTPUT knob clockwise. This increases the sweep current through the internal modulation coil coupling a magnetic field to the SQUID and the periodic $V-\phi$ curve appears on the CRO screen.

5. The voltage change that occurs due to the influence of magnetic field now appears on the vertical axis of the CRO. Observe the modulation depth at increased vertical sensitivity.

6. The maximum peak-to-peak voltage swing of the SQUID modulation $\Delta V$ is measured which is called the modulation voltage for the SQUID. A typical curve is shown in Figure 4.17.

![Image of $V-\phi$ curve for the SQUID with CH I at 0.2 V/div and CH II at 20 mV/div.](image)

Figure 4.17: $V-\phi$ curve for the SQUID with CH I at 0.2 V/div and CH II at 20 mV/div.

7. The BIAS OFFSET can be adjusted to maximize the modulation depth.

8. Since this is an amplified signal so divide the $\Delta V$ value (obtained from the CRO) by 10,000 to get the actual magnitude of the voltage swing.

9. Adjusting the FLUX OFFSET control enables us to view a specific region of the $V-\phi$ curve. It actually allows us to apply a static magnetic field on
the top of the oscillating field applied using the SWEEP OUTPUT. Turning this knob thus moves the V–ϕ curve left or right and enables us to explore points along the V–ϕ curve.

Q 13. Using the V–ϕ curve, find the mutual inductance of the SQUID chip with the internal modulation coil. HINT: Mutual inductance between the internal coil and the SQUID, $M_{int}$, is expressed as the ratio of the magnetic flux threading coil 2 produced by current flowing through coil 1.

$$M_{int} = \frac{\phi_2}{I_1} = \frac{\phi_0}{\Delta I_{period}} \quad (4.25)$$

where $\Delta I_{period}$ is the current required by the flux $\phi$ to complete 1 waveform. Refer to the description found in [3], Section 4.4, page no. 27-28.

Q 14. Discuss the function of the FLUX OFFSET control knob in the V – I and V–ϕ mode. Refer to page no. 17 and 23 of [3].

Resistance vs. temperature of the YBCO SQUID

Objective

This experiment helps us track the normally resistive to superconductive transition of the YBCO film that forms the SQUID in Mr. SQUID.

Apparatus

1. A commonly available silicon diode e.g., 1N914.
2. 200 cm long insulated copper wire (magnet wire).
3. A digital voltmeter (DVM) with sub millivolt resolution.
4. A soldering iron and electronics-grade solder.
5. A binder clip (stationery item).
7. Teflon tape.
8. DC constant current source of 10μA, constructed from the following equipment
• An operational amplifier e.g., 741.
• A zener diode 2.5-7 volts.
• A selection of resistors in the range 1 kΩ through 100 kΩ.
• A capacitor in the range 100 pF to 100 nF.
• Two 9-volt batteries.
• A solderless breadboard.

Constructing a constant current source (10µA)

In this part of the experiment, we will use a silicon diode as a thermometer. A constant current is supplied through the diode, operating in the forward bias region and the voltage drop across it is measured. The voltage is related to the temperature, as we show in the following discussion.

Q 15. Why does the resistance of a semiconductor decrease with increase in temperature? Consider the response of electron hole pairs to changes in temperature.

A simple constant current source can be build using a zener diode, two resistors, a capacitor, a general purpose operational amplifier and two 9 volt batteries. Build the circuit shown in Figure 4.18. The specific selection of the resistor $R_{\text{current}}$ is determined by the zener diode voltage. The circuit is designed in such a way so that the ratio of the zener diode voltage to the output resistance $R_{\text{current}}$ gives the value of the constant current generated by the circuit. Choose the components such that the output current is nearly 10µA.

![Circuit diagram for the constant current source.](image)

Q 16. Describe the working of the constant current source.
4.4  EXPERIMENTS

Silicon Diode as a temperature sensor

1. We will use the commonly available silicon diode to measure the temperature of the SQUID inside the probe because the voltage across the diode varies almost linearly with temperature.

2. This diode voltage as a function of temperature is given by the following equation

   \[ V_i(T) = \frac{E_g}{2q} - \left[ \ln(\alpha) + \frac{3}{2} \ln(T) - \ln(i_f) \right] \frac{k_B T}{q}. \quad (4.26) \]

   where

   \[ \alpha = \frac{1}{4} \left( \frac{2m k_B}{\pi h^2} \right)^{3/2} \frac{A k_B}{\tau E}. \]

   (The details are provided in [3] page no. 62, however there is a calculation error in [3].)

   **Q 17.** Derive Equation 4.26 by using the diode equation [4] for the forward bias current,

   \[ i_f = i_s \exp \left( \frac{q V_i}{k_B T} \right), \]

   where \( i_s \) is the saturation current through a \( pn \) junction semiconductor diode, \( q \) is the charge of carriers, \( T \) is the absolute temperature, \( k_B \) is Boltzmann's constant and \( V_i \) is the voltage across the diode.

   **Q 18.** Using Equation 4.26, plot \( V_i \) as a function of temperature \( T \).

3. After trimming the leads of the diode, solder the copper wires to its ends.

4. Connect the diode leads to the 10 \( \mu \)A DC constant current source and the DVM as shown in Figure 4.19. Use at least 50 cm of copper wire between the diode and the DVM.

5. Now turn on the current source and the DVM. There should be a voltage drop of approximately 0.3-0.4 V across the forward biased diode.

Temperature calibration

1. We can now calibrate the temperature response of the diode since the voltage drop across the diode increases almost linearly with decreasing temperature.

2. Note the temperature of the laboratory with the help of a thermometer and record the voltage drop across the diode at its temperature. The room temperature can be the first calibration point for the diode sensor.
3. Now slowly lower the diode right to the bottom of the dewar, three-fourth filled with $LN_2$. After the voltage reading has stabilized, again record the voltage across the diode. This voltage will lie somewhere between 0.9 and 1.1 volts and serves as the second calibration point, at 77 K.

4. A linear diode requires two calibration points, but the diode may be placed in an ice bath to get a third point also.

5. Now plot the diode voltage as a function of temperature and draw the line of best fit to the acquired data. In this way we have produced a calibration curve that interconverts diode voltage to temperature. Hence we have constructed a cryogenic temperature sensor using a Si-diode, a constant current source and voltmeter.

**Procedure for detecting the superconducting phase transition**

1. After performing the calibration we are required to plot the $R$-$T$ curve for the SQUID. We will now use the diode (attached with long copper wires) to sense the temperature in the vicinity of the SQUID chip and calculate the corresponding resistance by measuring the slope of the SQUID's $V$–$I$ curve which keeps on changing while reducing temperature.

2. For this purpose first remove the magnetic shield of the probe carefully, keeping the screw in a safe place to be recovered later.

3. We will temporarily mount the diode with the probe using the teflon tape as shown in Figure 4.20.

4. The diode is mounted to the back side of the chip. This process should be carried out very carefully so that the probe is not damaged in any case. Also avoid sticking teflon tape to the front side of the chip. DO NOT use ordinary tape to mount the diode.
5. Now put the magnetic shield back on its place with the help of its screw. The copper wires should trail down the bottom of the shield.

6. Use small cotton stuffing to close the bottom of the probe in order to improve the temperature uniformity.

Figure 4.20: Setup for diode mounting. (a) Diode attached to Mr. SQUID. (b) Diode is attached with the teflon tape. (c) Shield the probe with the copper wires trailing downward.

7. Now empty the dewar until there is about 10-12 cm of LN$_2$ left at the bottom and lower the probe into the dewar.

8. The binder clip along with the foam cover is used to prevent the probe from sliding down right to the base of the dewar. Start with the SQUID probe at the very top of the dewar as shown in figure 4.21.

9. Now connect the SQUID probe to the electronic box and the current and voltage outputs of the electronic box to the CRO. Turn the mode switch to the V – I position.

10. Connect the leads of the diode (coming from the bottom of the dewar) to the current source and DVM.

11. A straight line will be seen on the CRO screen, the slope of which gives the resistance of Mr. SQUID. At near room temperature (probe’s chip end
near the top end of dewar) it should be several hundred ohms.

12. Measuring resistance as a function of $T$

- When the voltage across the mounted diode stabilizes, record its value. Wait patiently for the DVM to show you a stable value of voltage drop across the diode and then measure the corresponding slope.
- Calculate and record the slope of the $V-I$ curve near the origin ($V=0$).
- Holding the probe, carefully loosen the binder clip and lower the probe about 1 cm further down into the dewar and record the voltage as well as slope of the $V-I$ curve.
- Repeat this procedure until the probe's chip end is completely immersed in $LN_2$ and attained the temperature of 77 K. The diode's voltage should correspond to the $LN_2$'s temperature at this point.

13. The expected transition of the slope while cooling the SQUID to $LN_2$'s temperature is shown in Figure 4.22. The $R-T$ graph should appear similar to Figure 4.23.

Figure 4.22: Transition of slope (resistance) from a finite value to zero during cooling to liquid nitrogen temperatures.
**Figure 4.23:** A typical $R - T$ graph.

**Q 19.** Plot the $R$-$T$ curve and note down the transition temperature of the YBCO film. Is the superconducting transition sudden? Is the resistance of the superconductor really zero?

**Q 20.** How does the resistance of a normal metal such as copper or silver change with temperature? What happens to the resistance at 0 K?

**Analog flux-locked loop (FLL)**

Mr. SQUID can be used as a sensitive magnetometer when employed in the so-called flux-locked loop (FLL) configuration. We know that in the resistive mode, the voltage across the SQUID is a sinusoidal function of the applied magnetic flux, with a period of one flux quantum $\phi_0$, but this is not the limiting resolution for the flux measurement. We can, quite accurately measure flux changes that are much smaller than a flux quantum. This extremely high sensitivity, makes the SQUID the most sensitive magnetometer!

**Objective**

In the present experiment, we will learn how the external coil of Mr. SQUID is used in a negative feedback loop for maintaining a constant magnetic flux through the SQUID ring. In the process, we will detect and measure ultra-small magnetic fields.
Principle of operation of the FLL

The FLL circuit is schematically represented in Figure 4.24. Here is the basic principle of operation.

![Flux-locked loop circuit diagram](image)

Figure 4.24: The flux-locked loop circuit, a schematic representation.

1. The SQUID (in the V – I mode) is biased with $i > I_C$.

2. The V–Φ mode is switched on and a sweep current is applied across the inner modulation coil of the SQUID. This induces a magnetic flux in the SQUID loop which results in a periodic voltage across it.

3. This periodic voltage is then fed into the input of the FLL circuit, where it is amplified and inverted. The resultant negative voltage is used to drive the external coil of the SQUID.

4. By using the gain of the inverted amplifier and adjusting the variable resistance of the potentiometer, (shown in Figure 4.24), the FLL is set up in such a way that the current flowing through the external coil creates a magnetic flux of opposite polarity and equal magnitude to the applied flux produced by the inner coil.

5. Thus two fluxes cancel the effect of each other and the SQUID will be in a zero magnetic flux state: it will be locked in a zero-flux condition.

6. By measuring the current being used to generate the opposing flux through the external coil and using the mutual inductance of the external coil to the SQUID, $M_{ext}$, we can determine the magnitude of the applied unknown magnetic flux.
7. The FLL technique is based on the principle of the conversion of magnetic flux, which is hard to measure, into voltage, which is easier to measure.

Figure 4.25: Copper wire connection to the EXT. COIL terminals to find its the coil's resistance at liquid nitrogen temperature.

Q 21. The terminals of external coil at low temperature are provided at the bottom of the probe as shown in Figure 4.25. Bind an $\approx 50$ cm long copper wire to each terminal of the coil and us find the coil's resistance $R_{ext}$ at 77K.

**Apparatus**

1. Mr. SQUID and $LN_2$.
2. Oscilloscope.
3. Two dual-operational amplifiers (e.g., HA17458P).
4. Two 9-volt batteries.
5. One 10 kΩ potentiometer.
6. A selection of resistors in the range of 1 kΩ through 100 kΩ.
7. A selection of capacitors in the range of 0.001 μF through 1 μF.
8. A selection of BNC connectors, hook up wires and alligator clips.

9. Copper wire to connect to the external coil connections.

**Procedure**

1. Connect the FLL circuit on the bread board, as shown in Figure 4.26.

![Circuit diagram for the flux-locked loop circuit.](image_url)

2. Before setting up Mr. SQUID, remove the top cover of the MS-EB03 electronics box and move the MODE switch next to the BNC connector into the direct (DIR) position. In this position, the signal connected to the EXT INPUT BNC on the rear panel of the Mr. SQUID electronics box is directly coupled through a 100 mA fuse at location F1 to the external feedback coil on the Mr. SQUID chip. **Be sure to reset the MODE switch in the buffered (BUF) position after you are finished with the experiment.**

3. In the $V - I$ mode, turn the SWEEP OUTPUT to its minimum so that you can just see a point on the CRO screen. Using the BIAS OFFSET knob adjust this point at the knee of the $V - I$ curve and turn on the $V - \phi$ mode.

4. Turn the SWEEP OUTPUT knob to set the amplitude of the flux ($\phi$) to about $\pm 0.5$ flux quantum by adjusting the single waveform symmetrically on the x-axis of the oscilloscope about the origin. This is shown in Figure 4.27, the top left corner.
5. Connect the voltage output of the electronic box to the input of the FLL circuit and CRO's channel II to either test point TP1 or test point TP2 as required.

6. The outputs at the TP1 and TP2 should look similar to Figure 4.27 (a) and (b). Compare the phase shift and amplitudes of TP1 and TP2.

![Scope photos of signals at TP1 (left) and TP2 (right) without external feedback. (a)TP1 showing flux being applied to the SQUID, (b) Amplified and inverted signal at TP2, (c) TP1 with the feedback flux superimposed, (d) Left over flux negatively amplified at TP2.](image)

Figure 4.27: Scope photos of signals at TP1 (left) and TP2 (right) without external feedback. (a)TP1 showing flux being applied to the SQUID, (b) Amplified and inverted signal at TP2, (c) TP1 with the feedback flux superimposed, (d) Left over flux negatively amplified at TP2.

7. TP1 simply shows the $V-\phi$ output of Mr. SQUID as a magnetic flux of $1\phi_0$ threads the SQUID loop. We can suppress the DC offset at TP1 by ac-coupling into the CRO.

8. TP2 also shows the $V-\phi$ output of Mr.SQUID but magnified by a factor of -10 and with an extra DC offset from the FLL circuit’s potentiometer. DC-couple the CRO at TP2 to see this offset and adjust the potentiometer so that the TP2 signal doesn’t contain this DC offset anymore.

**Q 22.** Describe the working of the difference amplifier circuit shown in Figure 4.26. Calculate its gain?

9. Now the circuit is ready to lock the flux state of Mr.SQUID. This is done by carefully connecting the FLL circuit output to the BNC connector labeled
10. Reconnect the power to your flux-locked loop circuit. The outputs at the
TPs should now look like those in Figures 4.27 (c) and (d).

11. At this stage, TP1 should ideally would be a flat line if the FLL circuit
perfectly canceled the flux applied by the Mr. SQUID box. But our output
at TP1 Figure 4.27 (c) is slightly sloped indicating that the cancelation was
not perfect due to the noise of the SQUID and the noise of the electronics.

12. The amount of flux threading the SQUID loop (before the feedback is
superimposed) is,

\[ \phi_{SQ} = N L_{SQ} I_{SQ} + M_{int} I_{int}. \]  \hspace{1cm} (4.27)

where \( N \) is the number of turns of SQUID coil, i.e., 3/4 [3]. \( L_{SQ} \) is the
SQUID's self-inductance, \( I_{SQ} \) is the amount of biasing current flowing
through the SQUID loop, \( M_{int} \) is the mutual inductance of the internal
modulation coil and \( I_{int} \) is the current flowing through it. The mutual
inductance \( M_{int} \) was calculated in Equation 4.25.

Q 23. What is the value of the current \( I_{SQ} \), when the SQUID is biased
approximately at the knee of the \( V - I \) curve?

13. After providing feedback through the external coil, the net flux linking the
SQUID is

\[ \phi_{Net} = L_{SQ} I_{SQ} + M_{int} I_{int} + M_{ext} I_{ext} \]
\[ = \phi_{SQ} + M_{ext} I_{ext} \]
\[ = \phi_{SQ} + \phi_{FB}. \]  \hspace{1cm} (4.28)

where \( \phi_{FB} \) is the negative feedback flux provided through the external coil.
Ideally \( \phi_{Net} \approx 0 \), i.e., \( \phi_{FB} = -\phi_{SQ} \).

14. The values of the relevant inductances and resistances are given in Table
4.1.

15. Calculate \( I_{ext} \) by measuring the voltage at TP2.

Q 24. Using the ideas developed above, estimate the efficacy of the
FLL, i.e., how much flux is the circuit actually canceling out.

Q 25. Describe how the FLL is actually working as a magnetometer.
What is the magnitude of the applied flux \( \phi_{SQ} \) before the feedback is ap-
pied? Calculate the uncanceled flux as a fraction of \( \phi_{0} \).
<table>
<thead>
<tr>
<th>Component</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_{int}$</td>
<td>Calculate from Q. 13, Section 4.4</td>
</tr>
<tr>
<td>$M_{ext}$</td>
<td>35 pH</td>
</tr>
<tr>
<td>$L_{SQ}$</td>
<td>73 pH</td>
</tr>
<tr>
<td>$R_{lb}$</td>
<td>2400 Ω</td>
</tr>
<tr>
<td>$R_{ext}$</td>
<td>Calculate from Q. 21, Section 4.4</td>
</tr>
</tbody>
</table>

Table 4.1: Values of Inductances and Resistances required for flux calculations
Chapter 5

Chasing Chaos with an RL-Diode Circuit

Junaid Alam and Sabieh Anwar

Have you ever wondered how science can be so organized and streamlined and yet be able to explain almost everything that we encounter? The answer to this genius skepticism is quite straightforward: we made it simple, for it was meant to simplify and systematize what seemed to be erratic and complicated. Science is a step towards intellectual sophistication in order to make things simple and explicable. This is what Leonardo da Vinci says, “simplicity is the ultimate sophistication.” In this experiment, on the contrary, we are going to shun the struggle for simplicity. Instead we are going to look into the complicated side of matters—those that appear to be simple. Let’s forget for the time being that simplicity is what we are after; let’s pursue complexity and intrigue; let’s chase Chaos.

KEYWORDS

Dynamical System · Supersensitivity · Phase Portrait · Poincare Map · Attractor · Fractals · Self-similarity · Feigenbaum Constant · Diode Recovery Time · Junction Capacitance · Resonance · Period Doubling Bifurcation · Chaos.

APPREHIMATE PERFORMANCE TIME 1 week.
5.1 Objectives

In this experiment, we will discover:

1. how very simple systems can exhibit complex behavior under certain conditions,

2. the richness of the mathematical and physical structure of dynamical systems,

3. how an arbitrarily small change in the input can change the long-term conduct of a dynamical system drastically,

4. how to construct and interpret phase portraits and Poincare Maps for different kinds of responses of a system,

5. the mystery of Feigenbaum constant and what makes chaos a universal underlying structure of the complexity exhibited by nonlinear dynamical systems,

6. a beautiful and artistic aspect of science in the form of attractors and fractals.
Bibliography


5.2 Foundations

**Summon up:** What kind of nonlinear phenomena have you come across? Try to list a few, with a reason to why you believe them to be nonlinear.
5.2. FOUNDATIONS

**Ponder:** Is it that all the linear processes that you have ever known about, are really linear? Remember one, and put it to test.

Mathematical linearization of scientific problems for the sake of practicality has been, in fact, a human confession that indicates a mere discrepancy regarding our reach and grasp over nature. It would be naïve to think that linearization works very often. We can hardly find linear processes in nature. The spirit of nature is indeed non-linear. On the other hand, no wonder, dynamical systems, systems having different behavior at different instants of time, comprise the core of scientific study. Therefore, in physics, nonlinear dynamical systems remain a vitally important subject. Our current experiment is about one such system.

**Defining Nonlinear Dynamics**

Nonlinear dynamics, is the field of physics and mathematics that deals with the most common kind of natural systems, systems that keep changing with time and are nonlinear. Being a bit more technical, dynamical systems for which the principle of superposition doesn’t hold are termed as nonlinear. For such systems the sum of responses to several inputs cannot be treated as a single response to the sum of those all inputs. Qualitatively speaking [2]:

A nonlinear system is a system whose time evolution equations are nonlinear; that is, the dynamical variables describing the properties of the system (for example, position, velocity, acceleration, pressure, etc.) appear in the equation in a nonlinear form.

Now, if \( x \) represents an input variable and \( y \) is the output as a function of \( x \), the principle of superposition in its very simplistic form states that:

\[
y(x_1 + x_2 + \ldots + x_n) = y(x_1) + y(x_2) + \ldots + y(x_n)
\]  

(5.1)

The above mathematical expression means that if the stimulus to a linear system is doubled, the response is also doubled. For a nonlinear system, the response will be greater or less than that.

**Ask yourself:** Could it be that a system is both linear and nonlinear at the same time? Can a system be linear for some conditions and nonlinear for others?
Nonlinearity: A Conduit to Chaos

What makes this nonlinearity so important? The basic idea is that for a linear system, when a parameter (e.g. the spring constant $k$ in a spring mass system) is varied, it doesn’t change the qualitative behavior of the system. On the other hand, for nonlinear systems, a small change in a parameter can lead to sudden and dramatic changes in both the qualitative and quantitative behavior of the system. For one value, the behavior might be periodic. For another value only slightly different from the first, the behavior might be completely aperiodic.

**Contemplate:** What could be a physical explanation of this unpredictability and sudden change in behavior?

Chaos defined

In the context of nonlinear dynamical systems, chaos is a word used to describe the time behavior of a system that is aperiodic, and is apparently random or “noisy”. But, underlying this chaotic randomness is an order that can be determined, in some sense, by the very time evolution equations that describe the system. Even when it may sound paradoxical, such an apparently random system is in fact deterministic.

**Understand:** What can you comprehend from the statement “such an apparently random system is in fact deterministic”? What is meant by such determinism?

Origins of chaos

Now, let us briefly try to answer the question: what could be the origin of chaos in nonlinear systems? Chaotic behavior shows up in systems that are essentially free from noise and are also relatively simple, i.e. possess only a few degrees of freedom. This tells us that chaotic behavior actually depends on the physical aspects and the spatiotemporal properties of a nonlinear system.

Chaos ringing the door-bell

Usually, chaotic behavior doesn’t appear without informing us when it is about to come, adding to the beauty of this brave and organized disorder. Generally,
it all starts with a so called *period-doubling bifurcation*: system switches to a new behavior with twice the period of the original system at a particular value of a certain parameter. As the value of that parameter is further increased, successive bifurcations occur and the behavior of system takes a time period that is four times, then eight times and so on, finally ending in chaotic behavior. This makes the story of chaos eventful and, as we shall see later, universal.

**The math ingredient:** A dynamical system is expressed by its differential equations. What happens to the solution of the system equations when a bifurcation occurs?

### 5.3 Identifying Chaos

We need to recognize chaos by face, for we are going to encounter it several times in our experiment. So, now we will learn about some useful tools that can help us identify chaos.

**Time series**

The very first and basic tool is the time series. Knowing the fact that chaos implies aperiodicity, we can tell if the system is chaotic by investigating the "shape" of its time series.

![Time-series plots](image)

(a) Periodic Time Series  
(b) Chaotic Time Series

**Figure 5.1:** Time-series for periodic and chaotic behavior.

But not all of the aperiodic responses imply chaos; many signals might be aperiodic due to the statistical noise or the complexity of the system. Therefore, time series analysis should be carried out for several initial conditions that may in turn validate the presence of chaos by exhibiting "divergence of nearby trajectories", i.e. difference in subsequent trajectories for slightly different initial conditions.
Fourier spectrum

From our discussion about the time series of a chaotic system, we may surmise that like any aperiodic response, the response of a chaotic system will have a continuum of frequencies when seen in the Fourier domain. This is indeed the case, and is a very useful tool to recognize chaos.

![Fourier spectra of periodic and chaotic behavior.](image)

Still one should not forget that Fourier spectrum of a completely random response is also a continuum of frequencies, so, this tool is useful when system approaches chaos after passing through several bifurcations and we can, in some way, track them.

**Exploit:** When you know what chaos looks like, can you come up with a scheme to harness chaos and put it to practical use?

Phase portraits

The notion of state space (or phase space) is a very rich topic. It has a venerable history of being helpful in stability analysis as well as quantitative inspection of dynamical systems. The basic idea in using a state space model is that if we are provided with the knowledge of the state variables, the variables that represent the state of a system, for a particular scenario and the rate of change of those state variables, we can predict the behavior of the system in terms of those variables at any time. This leads to the idea of using conjugate variables (Fourier duals of each other, or the position and momentum variables of the system) as the canonical coordinates for a state space representation. So, for a periodic system that obeys the law of energy conservation (e.g. a pendulum), the state space plot (phase portrait) will be one closed loop for a particular set of initial conditions. For a chaotic system, there will be many distinct loops in a phase
portrait, showing that the system is aperiodic and does not approach a stable trajectory.

![Possible phase portraits of periodic and chaotic behavior.](image)

Figure 5.3: Possible phase portraits of periodic and chaotic behavior. Refer to main text for labeling of axis.

**Example of a simple pendulum**

Consider a simple pendulum having a small amplitude of oscillation (so that we can assume \( \sin \theta \approx \theta \)). Ignoring friction, it may be represented using Newton's second law by a normalized second order differential equation of the form:

\[
\frac{d^2 \theta}{dt^2} + \frac{g}{l} \theta = 0
\]

(5.2)

where \( \theta \) represents the angular position of the pendulum. The solution of this equation will be:

\[
\theta = \theta_0 \sin(\omega t + \phi)
\]

(5.3)

where \( \theta_0 \) is the maximum angular displacement. The first derivative of \( \theta \) is:

\[
\dot{\theta} = \theta_0 \omega \cos(\omega t + \phi)
\]

(5.4)

Now, from (5.3) and (5.4), writing an equation in terms of \( \theta \) and \( \dot{\theta} \) will give the parametric equation:
\[ \frac{\theta^2}{\theta_0^2} + \frac{(\dot{\theta})^2}{(\omega \theta_0)^2} = 1 \] (5.5)

which is evidently the equation of an ellipse with \( \theta \) on the horizontal and \( \dot{\theta} \) on the vertical axis and represents a periodic trajectory in the phase space. In this context, \( \theta \) and \( \dot{\theta} \) represent the canonical coordinates. Using the two coordinates, we can find the state of the system at any instant.

**Implicate:** Write down the equation of energy of a pendulum in terms of position and momentum variables indicated in the formalism above. What is the total energy in the system?

**Figure out:** What does a closed loop in phase space signify? What can we say about the energy contained in a system?

**A step ahead:** Draw the circuit diagram of an RLC circuit. Write down the differential equation of the system and identify the canonical coordinates.

### Poincaré sections

Another very useful way of analyzing the behavior of a nonlinear dynamical system is a Poincaré Section or Poincaré Map. The basic motivation behind making such a map is to reduce an \( n \)-dimensional system to an \( (n-1) \)-dimensional system, making the analysis easier and a bit more intuitive.

![Poincaré sections](image)

**Figure 5.5:** Poincaré sections for periodic and chaotic behavior.

Constructing a Poincaré map is simple: sample the phase portrait of the system stroboscopically [6].

For periodic behavior, Poincaré map will be a single point. For chaotic or aperiodic behavior, there will be many irregularly distributed points in the map.
5.3. **IDENTIFYING CHAOS**

**Iterate:** In a Poincare map, why is there a single point for periodic and a scatter of points for aperiodic behavior? Can you construct phase space trajectories from a given Poincare section?

**Bifurcation diagram**

A very beautiful way of expressing the behavior of a dynamical system over the entire range of a particular parameter is the bifurcation diagram.

![Bifurcation diagram](source: wikipedia.org)

Figure 5.6: Bifurcation diagram. (source: wikipedia.org)

It shows a correspondence between the parameter values and the resulting response of the system. Every bifurcation indicates a successive period doubling and the response branches off into two. In figure (5.6), as the control parameter $\lambda$ is varied over a certain range, the response $x_n$ takes different number of values: two values at the first bifurcation, four values at the second bifurcation, eight values at the third bifurcation and so on. The fuzzy bands indicate chaotic behavior. Also, one can observe the periodic bands within the chaotic ones, showing that chaos can suddenly vanish and give rise to certain higher order periods. This is mainly because of the fact that differential equations defining the system may abruptly switch from chaos to a definite set of solutions for a certain value of the control parameter.

**Identify:** What does the presence of dark contours within the chaotic bands of the bifurcation diagram indicate?
Universality of chaos

As we have already signaled, chaos is not a mere state of unpredictability and disorder. It also enjoys the repute of a deterministic and universal framework that makes the canvas of this subject even multihued. The interesting thing to notice is that the same kind of mathematical description is quite germane to entirely different classes of systems: from control systems and lasers to climatology and chemical reactions. Let us just briefly touch few of its most amazing qualitative and quantitative aspects.

Fiegenbaum constant

When we look at a bifurcation diagram, such as the one shown in figure (5.6), we can see the distances between successive bifurcations getting smaller and smaller in a geometric way (along the horizontal axis). This is what Fiegenbaum noticed: the ratio of differences of parameter values at which successive bifurcations occur is the same for all the splittings [2]. Mathematically speaking:

\[
\delta_n = \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n} \quad (5.6)
\]

where \( \lambda_n \) is the parameter value at which the \( n \)th bifurcation occurs. Moreover, this ratio converges to a particular value—called the Fiegenbaum constant—as \( n \) approaches infinity:

\[
\delta \equiv \lim_{n \to \infty} \delta_n = 4.669201 \ldots \quad (5.7)
\]

This constant indicates a very universal and a quantitative equivalence between apparently very different physical systems.

A step ahead: Observe the diagram in figure (5.6) closely. Can you find out a similar constant for vertical spacings between successive branches each time they bifurcate?

Attractors and fractals

An important manifestation of the fact that chaos is deterministic are attractors: a set of points (or trajectories) to which all other trajectories—that start from the
initial conditions lying within a region called the basin of attraction—approach, as the time goes to infinity. Looking at the accompanying figure, we can observe how trajectories remain within a certain region of state-space. This confinement of trajectories within a certain region for a particular set of initial conditions is what points toward the determinism in the chaotic behavior.

Attractors, in addition to their aesthetic appeal and tendency to provide us with information about the active degrees of freedom in a system, also determine the dynamical properties of the system's long-term behavior.

The discussion on attractors cannot go without mentioning one of the most aesthetically rich notions in mathematics, namely fractals, that actually link attractors with the universality of chaos. Fractals are self-similar and self-replicating geometrical structures (figure (8)) that occur in the state space as attractors with a noninteger dimension and are sometimes called strange attractors. Noninteger dimension refers to the idea that, in general, these geometrical figures do not have a point, axis or plane of symmetry, and yet they are self-similar within themselves: they look the same at any degree of magnification. This is another...
characteristic beauty of chaos.

**Be intuitive:** Can you establish a connection between self-similarity of fractals and the universality of chaos as exhibited through Feigenbaum constant?

**The genius hunch:** What is the possible relationship between entropy and chaos?

### 5.4 The Experiment

A simple RL-Diode circuit is going to be the subject of this experiment. Although it is a simple system, it exhibits interesting behavior including bifurcations and chaos. A series arrangement will be used as shown in the figure below.

![RL-Diode circuit diagram](figure5_9.png)

**Figure 5.9:** The Experimental RL-Diode circuit [6].

#### The Circuit

The circuit (figure 5.9) will behave in two different modes: first when the diode is forward biased, the other when it is reverse biased.

![Diode forward bias and reverse bias](figure5_10.png)

**Figure 5.10:** Equivalent circuits for forward and reverse bias cycle.
5.4. THE EXPERIMENT

The Mathematical Model

**During the conducting cycle**, the circuit reduces to what is shown in figure (5.10a), with the diode acting as a fixed bias. The KVL expression turns out to be a first order differential equation, given as:

\[
L \frac{dI}{dt} + RI = V_0 \sin \omega t + V_f
\]  

(5.8)

where \(V_0\) is the peak amplitude of the AC input voltage and \(V_f\) is diode forward voltage drop. The solution of this equation, i.e. the current in the conducting cycle can be easily found out to be [4]:

\[
I(t; A) = \left( \frac{V_0}{Z_0} \right) \cos(\omega t - \theta) + \frac{V_f}{R} + Ae^{-Rt/L}
\]  

(5.9)

In equation (5.9), \(\theta\) represents the power factor angle or phase delay, given as:

\[\theta = \tan^{-1}(\omega L / R)\]

\(A\) is a constant of integration to be calculated using the initial conditions and \(Z_0\) is the forward bias impedance of the circuit and is equal to \(\sqrt{R^2 + \omega^2L^2}\).

**In the non-conducting cycle**, the diode behaves as a capacitor having a capacitance equal to its junction capacitance \((C_j)\). The equivalent circuit can be represented as a driven RLC circuit (figure (5.10b)). The loop equation for KVL will become a second order differential equation of the form:

\[
L \frac{d^2I}{dt^2} + R \frac{dI}{dt} + \left( \frac{1}{C_j} \right)I = V_0 \omega \sin \omega t
\]  

(5.10)

Equation (5.10) can be solved using the traditional two step technique of solving a non-homogeneous differential equation, i.e. separately for particular and homogeneous solutions.

**Derive:** Derive the solution of equation (5.10).

The final solution of equation (5.10) can be written as [4]:

\[
I(t; B, \phi) = \left( \frac{V_0}{Z_b} \right) \cos(\omega t - \theta_b) + Be^{-2Rt/L} \cos(\omega_B t - \phi)
\]  

(5.11)

The constants \(B\) and \(\phi\) are the constants of integration and can be found using the initial conditions of the cycle. Moreover, \(\theta_b\) is the phasor angle of the RLC network.
given as \( \theta_b = \tan^{-1}(L(\omega^2 - \omega_0^2)/R) \), \( \omega_0^2 = (1/LC) \) and \( \omega_0^2 = \omega_b^2 - (R/2L)^2 \).

**Exercise:** Instead of a piece-wise mathematical description, can you represent the circuit with a generalized differential equation?

**The Physical Model**

**The diode recovery-time**

Prior to looking into the practical behavior of the circuit and how it becomes chaotic, we need to understand the meanings and significance of an important parameter: the diode’s recovery time. The recovery time of a diode is the time a diode would take to completely stop the flow of forward current through itself as it moves into the non-conducting cycle. It depends on the amount of maximum forward current that has just flown through the diode. The greater the peak forward current, the longer the diode recovery time. Quantitatively speaking [4]:

\[
\tau_r = \tau_m[1 - \exp(-|I_m|/I_c)]
\]

(5.12)

where \( |I_m| \) is the magnitude of the most recent maximum forward current, and \( \tau_m \) and \( I_c \) are fabrication parameters for the specific diode.

**Bring to Light:** What can be a physical explanation of a diode’s junction capacitance? What relationship does it have with the recovery time?

**Route to Chaos**

A physical description of how the RL-diode circuit leads to period doubling is described in detail in [4].

When the circuit is operated at the resonant frequency, a certain amount of reverse current will flow through the diode in every reverse bias cycle due to the finite recovery time of the diode. If the peak current \( |I_m| \) is large in the conducting cycle (figure 5.11), interval ‘a’), the diode will switch off with a certain delay (figure 5.11, interval ‘b’) due to the finite recovery time and so will allow a current to flow even in the reverse-bias cycle. This reverse current, in turn, will prevent the diode from instantly switching on in the forward bias cycle; it will turn on with a delay (figure 5.11, interval ‘c’). This will keep the forward peak
5.4. **THE EXPERIMENT**

![Diagram](image)

Figure 5.11: Circuit Current and Diode Voltage \((\text{period-2})[4]\). The diode conducts when \(V_d = -V_f\) behaving as in the circuit in figure (10a). Otherwise it behaves as a capacitor as shown in figure (10b).

... current smaller than in the previous forward bias cycle, hence giving birth to two distinct peaks of the forward current. Notice that it took two cycles of the driving signal in this process. This is what we identify as a period-doubling bifurcation.

When the peak value of the drive voltage is increased, bifurcation to period-4 occurs, followed by higher bifurcations and eventually chaos. Figure 5.11 shows a period doubling scenario.

**Self-Assessment:** Briefly explain figure (5.11) according to the labels on the time axis, describing what happens at every marked instant.

**The Task**

**The Setup**

You will need a very simple and familiar set of components to deal with this experiment. The list of components is listed here.

1. Oscilloscope
2. Function Generator
3. Data Acquisition Setup
4. Bread Board
5. Circuit components

The Procedure

Now it is the time to start our experimental expedition:

1. Connect the components on the bread-board according to the circuit diagram.

2. Excite the circuit with a sinusoidal AC signal of minimum possible amplitude and a low frequency.

3. Observe the output of the circuit using the oscilloscope to find the resonant frequency of the circuit and the junction capacitance of diode.

4. Start increasing the amplitude gradually and observe the change in the time series plot of the output voltage. Note down the amplitude of the input at which the first bifurcation occurs.

5. Increasing the amplitude further, also observe and note down the input voltage amplitude at which higher bifurcations occur until chaos jumps in.

6. Repeat the measurements several times and calculate an average value of Feigenbaum constant from your data. Also observe the bifurcations while decreasing the input amplitude and hence find if there is any hysteresis.

**Expose:** What could be the possible reasons of hysteresis in this specific context?

7. By using the oscilloscope in the XY mode, identify period-doublings and calculate again the value of Feigenbaum constant.

8. Observe the Chaotic behavior in the XY plots and try to explain why chaos must have a fuzzy display.

**Ask Yourself:** Can you make a connection between the XY plots and the phase portraits?

**Observe:** Even in the chaotic behavior, there are several darker loops visible in the XY plots of scope. What can you tell about them?

9. Now, turn on the Computer, login and run the file RLD-DAQ.vi located on your desktop. Using this program, you can observe the circuit output and save the data to a text file for further processing.

10. By copying the data generated by LabVIEW into MATLAB, plot the phase
portrait for the circuit data obtained for periodic as well as chaotic regimes.

11. Using the known sampling frequency and the input frequency, plot the Poincare map for the circuit output data for several periodic and chaotic responses.

12. Using MATLAB, plot the Fourier spectrum for different kinds of behavior.

13. Repeat the procedure for output with different periods as well as for chaotic behavior.

**Ponder:** What do the peaks in the Fourier spectrum indicate? Why are there peaks even in the spectrum of chaotic output?

**Follow the Agents:** What are the characteristics of this particular circuit that make it exhibit chaos? Can you have several physical explanations?
Chapter 6

Phase Sensitive Faraday Rotation

Aysha Aftab, Rabiya Salman and Sabieh Anwar

Can light propagating through a medium be influenced by the application of an external magnetic field? You have observed optical activity in chiral molecules in your freshmen lab. The present experiment extends these concepts to magnetically induced birefringence through the historically important Faraday Effect, which reveals the rich interplay between optics and magnetism.

KEYWORDS
Polarization · Birefringence · Faraday rotation · Verdet constant · Phase-Sensitive Detection · Jones Calculus · Laser · Helmholtz coil · Resonance in RLC series circuit.

APPROXIMATE PERFORMANCE TIME 2 weeks.

PRE-REQUISITE EXPERIMENT: Basic measurements with the Lock-in amplifier.
6.1 Objectives

In this experiment, we will,

1. shed some light on the underlying mechanism of magnetically induced birefringence,

2. demonstrate the advantages of phase sensitive detection (PSD),

3. understand the mathematical formalism for polarized light and its manipulation,

4. build or use sources of uniform magnetic fields and measure the field strengths using a commercial magnetometer,

5. calculate numerical integrals,

6. build resonant RLC series circuit and understand the resonance phenomenon,

7. calculate the Verdet constant of terbium gallium garnet (TGG) and of a diamagnetic liquid.
Bibliography


6.2 Theoretical introduction

Q 1. What is polarization of light? Write down the equation for linear and circular polarization. Also, show that linearly polarized light can be written as a sum of left and right circular light [1]?

Magneto optical effect in transmission geometry

Michael Faraday observed the relationship between electromagnetism and light in 1845. Faraday's observation gave birth to the field of magneto optics: the interaction of optical radiation with magnetic media or the interaction of light with an optically inactive medium placed inside a magnetic field.

Birefringence

Some substances are optically anisotropic, i.e., their optical properties are direction dependent. An atom can be viewed as a positive charge surrounded by an electron shell with some binding forces (the dipole oscillator model).

For an anisotropic substance, the binding forces on the electron are anisotropic implying that the spring constant will be different in different directions: an electron displaced from its equilibrium position along one direction will oscillate with a different frequency than another direction. Since the electric field associated with light drives the electrons of medium at its frequency, these electrons re-radiate. The resulting secondary wavelets recombine and light propagates through a medium. The speed of the wave through the medium, is therefore, determined by the difference in natural resonating frequency of electrons and the frequency of the applied electric field. With anisotropy, the whole process becomes direction-dependent. Since the re-
fractive index, \( n = c/v \) is a function of speed, the anisotropy results in different refractive indices along different directions. This so-called 

\textit{birefringence} manifests itself the in rotation of the plane of polarization [1].

\textbf{Faraday rotation}

Chiral compounds exhibit rotation of linearly polarized light due to natural birefringence, but this birefringence can also be induced in otherwise optically inactive materials either by applying stress, magnetic or electric field. The Faraday effect is magnetically induced birefringence.

Linearly polarized monochromatic light while transmitting through an optically inactive material, under the influence of an axial magnetic field, is rotated by an angle \( \theta \) as shown in Figure 6.2. The angle of rotation \( \theta \) is given by,

\[
\theta = V B d.
\tag{6.1}
\]

provided the magnetic field remains uniform throughout the length \( d \) of sample.

For non uniform magnetic field, \( \theta \) is given by,

\[
\theta = V \int_0^d B(z)dz.
\tag{6.2}
\]

The proportionality constant \( V \) is a characteristic of the material, called the \textbf{Verdet constant} and is a function of the wavelength of light, temperature and refractive index of the material. It is the rotation per unit path length per unit applied magnetic field. In other words, it quantifies the induced birefringence. In this experiment you will measure this induced birefringence.
Larmor precession of the electron cloud in an applied magnetic field

We now try to posit some foundational arguments describing the underlying mechanism of Faraday rotation. Consider an electron, moving in a circle of radius \( r \) in a plane whose normal makes an angle \( \alpha \) with an applied magnetic field \( \mathbf{B} \). Since an electron is negatively charged its angular momentum \( \mathbf{L} \) and magnetic moment \( \mu_e \) are opposite to each other. The magnetic field exerts a torque \( \mathbf{T} \) on the magnetic dipole \( \mu_e \).

\[
\mathbf{T} = \mu_e \times \mathbf{B} = \mu_e B \sin \alpha.
\]

**Q2.** Referring to Figure 6.3 what is the direction of the torque on the magnetic dipole?

According to Newton’s second law, an angular impulse \( \mathbf{T} \) produces a change in angular momentum,

\[
\mathbf{T} \, dt = d\mathbf{L}.
\]

Thus, the attached vector \( \mathbf{L} \) rotates in anticlockwise direction. The resulting precession traced out by tip of the vector \( \mathbf{L} \) is shown in Figure 6.3. The angle of rotation through which angular momentum’s projection along the applied field, \( \mathbf{L}' \), moves in time \( dt \) is.

![Diagram](image)

**Figure 6.3:** Precession of angular momentum vector about the direction of the applied magnetic field.
\[
\phi' = \frac{dL}{L'} = \tau dt/L \sin \alpha
\]

and the precessional or the Larmor angular velocity becomes,

\[
\omega_L = \frac{d\phi}{dt} = \frac{\tau}{L \sin \alpha} = \frac{\mu_e B \sin \alpha}{L \sin \alpha} = \frac{\mu_e B}{L}.
\]  \hspace{1cm} (6.3)

The magnetic moment of circular current is given by

\[
\mu_e = iA = i(\pi r^2).
\]  \hspace{1cm} (6.4)

where,

\[
i = \frac{\epsilon \omega}{2\pi},
\]  \hspace{1cm} (6.5)

whereas the angular momentum of electron is given by,

\[
L = r \times p
\]

\[
L = mvr = mr^2 \omega.
\]  \hspace{1cm} (6.6)

Substituting Eqs (6.4), (6.5), (6.6) into (6.3), we get,

\[
\omega_L = \left( \frac{\epsilon \omega}{2\pi} \right) \left( \frac{\pi r^2}{mr^2 \omega} \right) B = \frac{eB}{2m}
\]  \hspace{1cm} (6.7)

showing that the Larmor frequency \( \omega_L \) is independent of the orientation of the current loop and the overall effect is the rotation of electronic structure about the direction of applied magnetic field \( [3] \).

**Semi-Classical description of induced birefringence**

You must have realized from Q 1 that plane polarized light is a combination of left and right circular (\( l \) and \( r \)) polarized light. Now, if light of vacuum frequency \( f \) is traveling through a medium whose electrons are rotating at the Larmor frequency then the \( l \) and \( r \) components will rotate the electron clouds with frequencies \( f + f_L \) and \( f - f_L \). Therefore in the dispersive medium, (refractive index is frequency dependent,) the functional dependence of the respective refractive indices can be written as,

\[
n_l = n(f - f_L)
\]

and

\[
n_r = n(f + f_L).
\]
If plane polarized light traverses a distance $d$ then the optical path lengths for $l$ and $r$ light are $n_l d$ and $n_r d$ respectively, so the optical path difference is $(n_r - n_l)d$. The difference of two refractive indices, the induced birefringence is,

$$n_r - n_l = n(f + f_L) - n(f - f_L)$$

Using the Taylor Series,

$$n_r - n_l = (n(f) + \frac{dn}{df} f_L) - (n(f) - \frac{dn}{df} f_L)$$

$$= 2f_L \frac{dn}{df}. \quad (6.9)$$

From Equation 6.8,

$$f_L = \frac{\omega_i}{2\pi} = \frac{eB}{4\pi m}$$

Eq (6.10) becomes,

$$n_r - n_l = 2(\frac{eB}{4\pi m} \frac{dn}{df}). \quad (6.10)$$

Figure 6.4: Superposition of left and right circularly polarized light into linearly polarized light. (a) Entering the sample, both the $l$ and $r$ components are moving with same speed and (b) while passing through the sample, these components have travelled with different velocities.

Since, phase change of a wave is $k$ ($= 2\pi/\lambda$) times the physical path traversed by the wave, the phase change for the two components is,

$$\phi_l = \left(\frac{n_l d}{\lambda}\right)(2\pi) \quad (6.11)$$

$$\phi_r = \left(\frac{n_r d}{\lambda}\right)(2\pi). \quad (6.12)$$
When \( l \) and \( r \) waves enter the sample, the phase difference is zero, but the phase difference accumulates as light passes through the sample. The vector sum of the two electric fields on emerging from the sample is shown as \( \mathbf{E} \) with a net rotation \( \theta \) from its initial value. Since, \( \mathbf{E} \) is an equal superposition of \( l \) and \( r \) components, we see from Figure (6.4) that,

\[
\phi_l - \theta = \phi_r + \theta \\
\Rightarrow \theta = \frac{\phi_l - \phi_r}{2}.
\]

Thus, the Faraday rotation angle is,

\[
\theta = \frac{1}{2} \left( \frac{2\pi d}{\lambda} \right) (n_l - n_r) \\
= \left( \frac{\pi d}{\lambda} \frac{eB}{2\pi m} \right) \left( \frac{dn}{df} \right) \\
= \frac{e}{2m\lambda} \left( \frac{dn}{df} \right) B d.
\]

Comparing Eq (6.1) and (6.13), the Verdet constant,

\[
V = \frac{e}{2m\lambda} \left( \frac{dn}{df} \right)
\]

which is a function of wavelength and the dispersion [3]. The Faraday rotation is a direct result of \( n_l \neq n_r \), arising because of the frequency dependent refractive index.

**Description of dispersion based on the Zeeman Effect**

The physical reason behind the change in refractive index can also be explained through Zeeman splitting. The splitting of spectral lines of the atom when placed in magnetic field is called the Zeeman effect, after the name of discoverer P. Zeeman. From Eq (6.3),

\[
\mu_e = \frac{L\omega_L}{B} = \left( \frac{L}{B} \right) \left( \frac{eB}{2m} \right) = \frac{eL}{2m}
\]

Since, the left and right (\( l \) and \( r \)) components of light carry an angular momentum of \( +h \) and \( -h \) respectively, the \( l \) component drives electron into left circular motion and the \( r \) component drives electrons into right circularly motion, resulting in different magnetic moments. Interaction of the magnetic moment \( \mu_e \) with the magnetic field \( \mathbf{B} \) slightly shifts the energy of atomic level by an amount

\[
\Delta E = -\Delta(\mathbf{\mu_eB}) = -(\Delta \mu_e) B = -\left( \frac{e}{m} \right) B.
\]
Thus, under the application of an axial magnetic field, dispersion curves for left and right circularly polarized light are identical but displaced by the frequency difference between the two Zeeman components.

\[ \Delta \omega = \frac{\Delta E}{\hbar} = -\frac{e}{m} B \]

which results in two different refractive indices \( n_r \) and \( n_l \) and therefore a different speed, at a given \( \omega \) [2].

![Graph showing refractive indices for left and right circularly polarized components of plane wave in the presence of magnetic field. The dispersion curves for the two components are shifted by \( \Delta \omega \)](image)

**Jones calculus**

Jones calculus, invented by the American physicist R. Clark Jones, in 1941, is a useful formalism to understand the state of polarization of perfectly polarized light as well as its transformation by various optical devices. For example, polarized light given by,

\[ \mathbf{E}(z,t) = \hat{\mathbf{x}} E_{ox} \cos(kz - \omega t + \phi_x) + \hat{\mathbf{y}} E_{oy} \cos(kz - \omega t + \phi_y) \]  

(6.17)

is represented in the Jones formalism as,

\[ \begin{pmatrix} E_x(z,t) \\ E_y(z,t) \end{pmatrix} = \begin{pmatrix} E_{ox} e^{i \phi_x} \\ E_{oy} e^{i \phi_y} \end{pmatrix} e^{i(kz - \omega t)}. \]  

(6.18)

The two component column vector completely specifies the amplitude and phase of electric field and hence its state of polarization. This is called the Jones vector. Most of the times, it is not necessary to know the exact phase but the phase difference \( \epsilon = \phi_y - \phi_x \) between the \( x \) and \( y \) components. Moreover, \( e^{i(kz - \omega t)} \) is
always understood to be present. Accordingly, Jones vector can also be written as,

$$\hat{\mathbf{E}}(z, t) = \begin{pmatrix} E_{ox} \\ E_{oy} e^{i\epsilon} \end{pmatrix} e^{i\phi_v}.$$  \hfill (6.19)

Ignoring the term $e^{i\phi_v}$

$$\tilde{\mathbf{E}}(z, t) = \begin{pmatrix} E_{ox} \\ E_{oy} e^{i\epsilon} \end{pmatrix}.$$  \hfill (6.20)

For linearly polarized light $\epsilon = 0$ or $180^\circ$, therefore the general form of Jones vector for linearly polarized light is,

$$\tilde{\mathbf{E}}(z, t) = \begin{pmatrix} E_{ox} \\ E_{oy} \end{pmatrix}.$$  \hfill (6.21)

Jones vectors can be normalized such that the sum of the squares of their components is 1, i.e.

$$E_{ox}^* E_{ox} + E_{oy}^* E_{oy} = 1.$$  

This normalized form discards the amplitude information needed for absorption calculations, but simplifies analysis in many other cases. The normalized form of (6.21) at an angle $\alpha$ w.r.t an arbitrary reference axis is,

$$\tilde{\mathbf{E}}(z, t) = \begin{pmatrix} \cos \alpha \\ \sin \alpha \end{pmatrix},$$

where, the angle $\alpha$ is defined such that,

$$\cos \alpha = \frac{E_{ox}}{\sqrt{E_{ox}^2 + E_{oy}^2}} = E_{ox},$$

$$\sin \alpha = \frac{E_{oy}}{\sqrt{E_{ox}^2 + E_{oy}^2}} = E_{oy}.$$ 

Q 3. Write down the normalized Jones column vector for horizontally, vertically, left and right circularly polarized light?

Suppose that the Jones vector for polarized incident beam $\hat{\mathbf{E}}$, is represented by $\tilde{\mathbf{E}}$, after transmission through an optical element then, the optical element can be represented as a $2 \times 2$ transformation matrix $J$, called the Jones matrix, given by

$$\tilde{\mathbf{E}}_r = J \tilde{\mathbf{E}}.$$  \hfill (6.22)
where

\[ J = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}. \] (6.23)

Equation 6.22 can be written as,

\[ \begin{pmatrix} \bar{E}_{nx} \\ \bar{E}_{ny} \end{pmatrix} = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix} \begin{pmatrix} E_{nx} \\ E_{ny} \end{pmatrix}. \] (6.24)

If the beam passes through a series of optical elements represented by the matrices \( J_1, J_2, J_3, \ldots, J_n \), then

\[ \bar{E}_f = J_n \cdots J_3, J_2, J_1 \bar{E}_i. \] (6.25)

The matrices do not commute, so they must be applied in proper order.

**Q 4.** Show that the transformation matrix \( J_h \) for horizontal linear polarizer is

\[ J_h = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \] (6.26)

### 6.3 Experimental Technique

**Why PSD in Faraday rotation?**

You have already performed an introductory experiment of using the lock-in amplifier, so without discussing the details of the technique and the instrumentation any further, we will only focus on why are we using phase sensitive detection (PSD) in this experiment. Consider a simple optical system used to measure the transmission of light through a medium. Let us suppose a small response ob-

![Figure 6.6: A simple optical system.](image)

scured by overwhelming noise is to be measured. The output signal in this case
will be,

\[ V_o = V_{\text{sig}} + V_{\text{noise}}. \]  

(6.27)

The noise and signal amplitudes for such a system as a function of frequency

![Graph showing noise and signal amplitude as a function of frequency](image)

Figure 6.7: (a) Noise and signal amplitude as a function of frequency. (b) Modulating the signal to a region of low noise.

are shown in Figure (6.7) [4]. The large peaks at 50 Hz and its multiples are due to electrical interference from the mains power lines. The noise power increases at lower frequencies (remember this is due to 1/f noise). Faraday rotation is extremely small in magnitude. If such a small signal buried in noise is to be measured, amplifying the signal will not improve the signal-to-noise ratio, the noise is amplified with the signal. A clever approach is to move the signal to a region of low noise, to higher frequency. For example, in the present experiment, we use an ac magnetic field for inducing Faraday rotation instead of a dc field produced by dc current or a permanent magnet. This technique gives two real advantages.

- The weak signal of interest buried in noise can be extracted successfully through PSD.

- Faraday rotation can be observed at smaller values of magnetic field (e.g., 80 G rms). This circumvents the need for large, expensive, bulky, water-cooled electromagnets for producing large magnetic fields.

Q 5. Can you think of a simple experiment that measures the noise spectrum of laser light detected by a photodetector?

Q 6. What is Malus’s law? How does a polarizer work?
Overview of the experiment

The plane of polarization of linearly polarized monochromatic light traversing through the sample $S$ of length $d$ placed under the influence of an ac magnetic field is rotated. Since the field is oscillatory, the rotation angle is also oscillatory. Another polarizer set at an arbitrary angle relative to input polarizer subsequent to the sample is required to analyze the rotation. The analyzer converts the polarization modulation to an amplitude modulation by the way of Malus’s Law. The emerging light beam carrying the information in the form of amplitude variations is incident upon a photodiode whose output appears in the form of current proportional to the light intensity.

Let us suppose incident light polarized along the $x$-axis is propagating in the $z$ direction. The electric field in terms of Jones vector is,

$$
\tilde{E}_o = \begin{pmatrix} 1 \\ 0 \end{pmatrix} A_o \exp(i(kz - wt))
$$

(6.28)

where $A_o$ corresponds to the amplitude of the electric field. Suppose, the analyzer is set at an angle $\phi$ w.r.t the polarizer. Jones transformation matrix for the analyzer is,

$$
J_{\text{ana}}(\phi) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}.
$$

(6.29)

**Q 7.** Write down the equation for electric field after transmission through analyzer and derive the matrix (6.29)?(HINT: Use Figure 6.8.)

After passing through the sample $S$ of length $d$ placed in magnetic field, the plane of polarization of light is rotated by an angle $\theta$, so the Jones vector after emerging from the sample is,
\begin{equation}
\begin{pmatrix}
\cos \theta \\
\sin \theta
\end{pmatrix}
\end{equation}
and the corresponding electric field is,
\begin{equation}
\mathbf{E} = \begin{pmatrix}
\cos \theta \\
\sin \theta
\end{pmatrix} A_0 \exp(i(kz - \omega t)).
\end{equation}

**Q 8.** Since, the analyzer is set at an angle $\phi$ w.r.t the polarizer, show that the electric field of the light beam after emerging from the sample followed by the analyzer is,
\begin{equation}
\tilde{\mathbf{E}} = \begin{pmatrix}
\cos(\phi - \theta) \cos \phi \\
\cos(\phi - \theta) \sin \phi
\end{pmatrix} A_0 \exp(i(kz - \omega t)).
\end{equation}

The intensity of light measured by the photodetector is,
\begin{equation}
I = k A_0^2 |\cos^2(\phi - \theta)|.
\end{equation}

**Q 9.** Derive the expression (6.33). What are the dimensions of the constant $k$? In the subsequent discussion, we will normalize $k = 1$. (HINT: Use the concept of the Poynting vector.)

**Q 10.** Write the Jones transformation matrix for the combination of the polarizer, sample and analyzer, placed in the same order.

**Optimization of the analyzer angle**

According to Eq. 6.33, the rotation of the plane of polarization manifests as a change in intensity at the photodiode. To get maximum change in intensity, the analyzer angle needs to be optimized. Differentiating the intensity w.r.t $\phi$, we
6.3. EXPERIMENTAL TECHNIQUE

get,

\[ \frac{dl}{d\phi} = A_0^2 \cos(\phi - \theta) \sin(\phi - \theta) \]  \hspace{1cm} (6.34)

\[ = A_0^2 \sin 2(\phi - \theta). \]  \hspace{1cm} (6.35)

Differentiating again,

\[ \frac{d^2l}{d\phi^2} = 2A_0^2 \cos(2(\phi - \theta)). \]  \hspace{1cm} (6.36)

Maximum change in intensity is obtained by maximizing \( \frac{dl}{d\phi} \) or by setting \( \frac{d^2l}{d\phi^2} = 0 \).

\[ 2A_0^2 \cos 2(\phi - \theta) = 0 \]

since, \( A_0 \neq 0 \), we have,

\[ \cos 2(\phi - \theta) = 0 \]

\[ (\phi - \theta) = 45^\circ. \]

Since, the Faraday rotation \( \theta \) is much smaller than \( \phi \), maximum \( \Delta l \) is obtained when the analyzer is set at \( 45^\circ \) relative to polarizer. The measured intensity is,

\[ I = \frac{A_0^2}{2} \left[ 1 + \cos 2(\phi - \theta) \right] \]

\[ = \frac{A_0^2}{2} \left[ 1 + \cos(2\phi) \cos(2\theta) + \sin(2\phi) \sin(2\theta) \right] \]

\[ = \frac{A_0^2}{2} \left( 1 + \sin(2\phi) \sin(2\theta) \right). \]

For \( \phi = 45^\circ \) and \( \sin(2\theta) \equiv (2\theta) \),

\[ I \approx \frac{A_0^2}{2} (1 + 2\theta). \]  \hspace{1cm} (6.37)

The field is made oscillatory, with an oscillating frequency \( \Omega \),

\[ B = B_0 \sin(\Omega t). \]

and since the angle of rotation is directly dependent on the magnetic field,

\[ \theta = \theta_0 \sin(\Omega t). \]  \hspace{1cm} (6.38)

therefore, Eq. (6.37) can be written as,

\[ I \approx \frac{A_0^2}{2} \left( 1 + 2\theta_0 \sin(\Omega t) \right). \]  \hspace{1cm} (6.39)

**Converting light intensities into photocurrents**

The photodiode converts the photon intensities into current, thus

\[ i = \dot{I} + \delta \dot{I}. \]
where \( i_{dc} = \frac{A_0^2}{2} \) and \( i_{ac} = \theta_0 A_0^2 \sin(\Omega t) \). Modulated photocurrent due to Faraday rotation \( i_{ac} \) is measured through lock-in amplifier which displays the rms values, therefore the output of the lock-in amplifier is,

\[
i_{ac} = \frac{i_{ac}}{\sqrt{2}} = \frac{\theta_0 A_0^2}{\sqrt{2}}.
\]

Taking the ratio of the modulated current (shown by the lock-in amplifier) to the steady current, we obtain,

\[
\frac{i_{ac}}{i_{dc}} = \frac{\theta_0 A_0^2}{\sqrt{2}} \Rightarrow \theta_0 = \frac{i_{ac}}{\sqrt{2} i_{dc}}
\]  

and as far as the rms value of the Faraday rotation angle is concerned,

\[
\theta_r = \frac{\theta_0}{\sqrt{2}} = \frac{i_{ac}}{2 i_{dc}}
\]

Figure 6.10: Signal measured by photodiode is made up of two parts, average light intensity, \( i_{dc} \) and modulated intensity at the frequency of ac magnetic field, \( i_{ac} \). The currents are proportional to the intensities.

\[
\theta_r = \frac{\theta}{\sqrt{2}} = \frac{i_{ac}}{2 i_{dc}}
\]

The dc component is measured by an oscilloscope in the absence of magnetic field while the ac component is measured by the lock-in amplifier in the presence of the magnetic field. For a uniform magnetic field Verdet constant is determined from the experimental values of \( \theta \), \( B \) and \( d \),

\[
\theta = VBd,
\]

whereas for non uniform magnetic field, \( \theta \), is given by,

\[
\theta = V \int_{0}^{d} B(z) dz.
\]

**Q 11.** What is the working principle of a photodetector? What does the photodetector measure? The electric field or the intensity?
Q 12. Can the photodiode measurement be affected by stray magnetic field? (HINT: the Hall Effect)

Schematic of the experiment

Figure 9.2 shows the schematic diagram of the experimental setup for the observation of Faraday rotation.

![Schematic diagram of experimental setup for Faraday rotation](image)

Figure 6.11: Schematic of experimental setup for Faraday rotation.

The setup comprises these components:

(a) Light Source

(b) Mechanism for producing and measuring an oscillating magnetic field

(c) Detection devices

Light source

Light from a lamp can be used after collimating it by a lens and passing through a color filter to make it monochromatic, however, since, LASER is a source of highly directional and monochromatic light and is easily available, it is convenient to use it as the light source. HeNe laser of wavelength 633 nm or an electrically pumped diode laser of wavelength 405 nm will be used in the experiment.
<table>
<thead>
<tr>
<th>Component</th>
<th>Supplier</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Light source</strong></td>
<td></td>
</tr>
<tr>
<td>laser 633 nm, 2 mW</td>
<td>Thorlabs (HRR-020)</td>
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<tr>
<td>laser 405 nm, 40 mW</td>
<td>B&amp;W TEK (405-40E)</td>
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<td><strong>Linear Polarizer</strong></td>
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<td>extinction ratio=1000 : 1</td>
<td>Thorlabs (LPV1050)</td>
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<td>GW-Instek (SFG-1013)</td>
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<td>CERWIN VEGA</td>
</tr>
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<td>Panasonic electronics</td>
</tr>
<tr>
<td>Helmholtz coil, 120 G rms</td>
<td>Homemade</td>
</tr>
<tr>
<td><strong>Detection element</strong></td>
<td></td>
</tr>
<tr>
<td>Photodiode</td>
<td>Newport (818-SL)</td>
</tr>
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<td><strong>Measuring instruments</strong></td>
<td></td>
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<tr>
<td>Lock in amplifier</td>
<td>Stanford Research System (SR-510)</td>
</tr>
<tr>
<td>Oscilloscope</td>
<td>GW-Instek</td>
</tr>
<tr>
<td>Gaussmeter with axial and transverse probes</td>
<td>LakeShore (410)</td>
</tr>
<tr>
<td>Clamp meter</td>
<td>Kyoritsu (KEW SNAP 2017)</td>
</tr>
<tr>
<td>LCR meter</td>
<td>QuadTech, Inc.</td>
</tr>
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<td><strong>Accessories</strong></td>
<td></td>
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<td>Thorlabs (RC1)</td>
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</tr>
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<td>Homemade</td>
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<td>Homemade</td>
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<tr>
<td>Teflon crystal holder</td>
<td>Homemade</td>
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<tr>
<td>M6 and M4 screws</td>
<td>Thorlabs</td>
</tr>
<tr>
<td>TGG crystal d=1 cm</td>
<td>Castech Inc.</td>
</tr>
</tbody>
</table>
Q 13. What is the basic principle of a laser? How does a HeNe laser work?

**Mechanism for producing and detecting the magnetic field**

In principle, both ac and dc magnetic field can be used in this experiment. Dc sources include permanent magnets or solenoids having steady current in their windings. Since, Faraday rotation is small in magnitude, of the order of microradians, so a large dc magnetic field, of several kilo-gauss will be required to achieve a sizeable rotation, which in turn requires large and bulky dc magnets or a large dc power supply to produce required field [8]. However, using an ac magnetic field, the rotation becomes oscillatory and can be tracked by PSD. For example, in this experiment, you will be provided with a Helmholtz coil capable of generating a field of approximately 120 G rms.

**The Helmholtz coil**

A pair of Helmholtz coils is used to produce a uniform magnetic field over a large volume of space. It consists of two identical coils such that separation \( d \) of the coils is equal to their common radius \( a \).

![Diagram of Helmholtz coils](image)

Figure 6.12: Magnetic field at point \( P \) due to single circular coil carrying a current \( i \).

Let us consider a single loop of conductor of radius \( a \), carrying a current \( i \). Using the Biot-Savart rule, magnetic induction at the point \( P \), at a distance \( r \) is,

\[
dB = \frac{\mu_0}{4\pi r^2} i \, dl \times \mathbf{u},
\]

(6.45)

where, \( \mathbf{u} \) is the unit vector connecting the conducting element with the point at which the field is to be determined, \( \mu_0 \) is permeability of free space = \( 4\pi \times 10^{-7} \).
Hm\(^{-1}\). This geometry is shown in Figure 6.12. Substituting,

\[ r = \frac{a}{\sin \alpha} \]  

(6.46)

into Eq (6.45), we get,

\[ dB = \frac{\mu_0}{4\pi a} (\sin^2 \alpha) i \, dl \times u. \]  

(6.47)

The axial component of magnetic induction is

\[ dB_{ axial} = dB \sin \alpha. \]

Therefore,

\[ dB_{ axial} = \frac{\mu_0}{4\pi a} (\sin^3 \alpha) i \, dl \times u \]  

(6.48)

Since, \(dl\) is perpendicular to \(u\), and integrating round the coil \(\int dl = 2\pi a\), we obtain the total axial field,

\[ B_{ axial} = \frac{\mu_0 i}{4\pi a^2} (\sin^3 \alpha) 2\pi a \]

\[ = \frac{\mu_0 i}{2a} \sin^3 \alpha \]

\[ = \frac{\mu_0 i}{2a} \frac{a^3}{(a^2 + z^2)^{3/2}} \]

\[ = \frac{\mu_0 i}{2a} \frac{a^2}{(a^2 + z^2)^{3/2}}. \]  

(6.49)

For \(N\) number of turns, Eq. (6.49) becomes,

\[ B_{ axial} = \frac{N \mu_0 i}{2} \frac{a^2}{(a^2 + z^2)^{3/2}} \]

\[ = \frac{\mu_0 N i}{2a}. \]  

(6.50)

For the Helmholtz pair, if one coil is placed at \(z = 0\) and the other at \(z = a\), and if current flows through both the coils in same direction (referred to as superposition condition, Figure (6.13)), by symmetry the radial component of magnetic field along the axis must be zero. Hence, the magnetic field on the common axis of the coils becomes [6],

\[ B = \frac{\mu_0 N i}{2a} [1 + \frac{z^2}{a^2}]^{-3/2} + [1 + \frac{(a-z)^2}{a^2}]^{-3/2}. \]  

(6.51)

Q 14. Show that at the point on the axis midway between the coils \(z = a/2\), the field is.

\[ B = \left(\frac{4}{5}\right)^{3/2} \left(\frac{\mu_0 N i}{a}\right). \]  

(6.52)

Q 15. Using the binomial expansion \((1+z)^n = 1 + nz + \frac{n(n-1)}{2!} + \ldots\), show that Eq. (6.51) can also be written as,

\[ B = \frac{\mu_0 N i}{2a} (1 + c_4 z^4 + c_6 z^6 + \ldots). \]  

(6.53)
Figure 6.13: Pair of Helmholtz coil with separation equal to the common radius and carrying the current in the same direction.

where, $c_4 = 15/8a^4$ and $c_6 = -105/48a^6$.

Q 16. What do you conclude from equation (6.53) about the uniformity of the magnetic field? How does the field in the middle of the coils compare with the field in the center of a single circular loop of the same radius?

Figure 6.14: Instruments for creating and detecting the oscillating magnetic field.

In our experiment, the Helmholtz coil is constructed from 18 gauge copper wire (diameter 1.2 mm). Each multilayer coil consists of 18 turns in 18 layers, the coil’s outer and inner diameters are 10.2 cm and 6.5 cm respectively. The length of each coil is 2.7 cm and radius is 4.5 cm. Inductance of the coils, determined
using the LCR meter, is found to be 7 mH with a resistance of 1.5 Ω for each coil, so the total inductance of the Helmholtz pair is 15 mH and the total resistance is 3 Ω. The Helmholtz coil pair constitutes a series $RLC$ circuit. At resonating frequency $\omega_r$, the inductive reactance $X_L$ is equal to the capacitive reactance $X_C$ and total impedance is purely resistive. The resonating frequency is,

$$\omega_r = \sqrt{\frac{1}{LC}}$$

or

$$f_r = \frac{1}{2\pi \sqrt{LC}}.$$

Q 17. Calculate the resonating frequency when a capacitor of 0.97 μF is connected in series with the coil? Why is the Helmholtz coil made resonating?

### 6.4 The Experiment

1. Assemble the setup according to Figure (9.2). Turn on the audio amplifier and the function generator. Amplify an approximately 1 V, 70 Hz sinusoidal signal through audio amplifier. Apply this amplified output to the Helmholtz coil.

2. Increase the frequency of the ac signal applied to the coil. Tabulate the frequency against current passing through the Helmholtz coil (Table 10.2) and plot the frequency response. The current is measured with the help of a clamp meter or an ammeter.

<table>
<thead>
<tr>
<th>Frequency (Hz)</th>
<th>Current (A rms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>.</td>
</tr>
<tr>
<td></td>
<td>.</td>
</tr>
<tr>
<td></td>
<td>.</td>
</tr>
<tr>
<td>2500</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.2: Mapping the frequency response of the Helmholtz coil.

3. Set the function generator at the resonating frequency. Increase the current by increasing the gain of the amplifier (Table 6.3). Measure the magnetic field using the Gaussmeter in ac mode (LakeShore, Model 410) equipped with transverse probe at the midpoint between the two coil. Plot a graph
between current and magnetic field. Do you observe a linear relationship as predicted by equation (6.52)?

<table>
<thead>
<tr>
<th>Current (A rms)</th>
<th>Magnetic field (G rms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>.</td>
<td></td>
</tr>
<tr>
<td>.</td>
<td></td>
</tr>
<tr>
<td>.</td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td></td>
</tr>
</tbody>
</table>

Table 6.3: Linear relationship between the current (A) and the magnetic field (Gauss).

*Next you will determine the Verdet constant for Terbium Gallium Garnet (TGG) crystal and a diamagnetic liquid such as carbon disulphide CS2.*

Never touch the lateral surfaces of the TGG crystal.

4. Turn on the provided laser. The HeNe takes about 30 mins to warm up and reach a stable value. *Never look directly at the laser light.* Always wear safety goggles when operating the laser. For the time being, keep the laser power switched on but close the shutter of the laser head.

5. The TGG crystal is 1 cm long but CS2 is filled in a 6 cm long glass cell. We cannot expect the magnetic field in between the coils of Helmholtz pair to be uniform over this large a distance. So you need to map the magnetic field profile and perform numerical integration of the magnetic field as suggested in Equation 6.44.

*The next few steps will help you calibrate the magnetic field.*

6. Fix a scale with the edge of the sample holder. Place the glass cell over the crescent shaped sample holder. Open the laserhead shutter. Adjust the height of the laser and sample holder to pass the beam through the center of the coils. You might need to adjust the heights of optical components by translating the stainless steel posts. Close the shutter again. The height of the sample holder will now be kept fixed. Mark the end points of the sample on the scale.

7. Remove the sample. Fix the axial probe on another holder on either side of the Helmholtz coil and turn on the Gauss meter. Select the 200 G range and ac mode. Switch on the audio amplifier and tune the function generator to the resonating frequency, 1.22 kHz. Current is now passing
through the Helmholtz coil. Select some value of current and measure the corresponding magnetic field at the midpoint between the Helmholtz coils (A field of 90 G rms is a reasonably good value). Move the probe away from the center of coils on both sides. Check that magnetic field is not reaching the polarizers and photodiode. If required, adjust the distances by moving the rail carriers along the length of the optical rail. Map the magnetic field profile by moving the probe along the length of the sample with a step size of 0.5 cm, for different values of current (similar to the observation tabulated in (11.4)). Remove the probe and turn off the magnetic field. Tabulate the variation in magnetic field along the length of the sample as in Table 11.4.

8. Estimate the numerically integrated magnetic field over the length of the sample \(d\).

\[
\Gamma = \sum_k B_k(z) \Delta z^k.
\]  

(6.54)

Table 6.4: Numerical integration of the magnetic field.

| \(i\) (A rms) | .1 | .2 | .3 | .4 | .5 | .6 | .7 | .8 | .9 | 1  | 1.1 | 1.2 | 1.3 | 1.4 |
| ------------- |----|----|----|----|----|----|----|----|----|----|----|----|----|
| \(z\) (cm)   |    |    |    |    |    |    |    |    |    |    |    |    |    |
| \(B_o\) (G rms) |    |    |    |    |    |    |    |    |    |    |    |    |    |
| \(\Gamma\) (G cm) |    |    |    |    |    |    |    |    |    |    |    |    |    |

You will now measure \(\dot{u}_c\), the dc component of the detected signal, in the absence of magnetic field, with the sample in place. Perform optical alignment if required.

9. Connect the photodiode to the oscilloscope, select the dc input mode. Remove the background reading from ambient light either by placing a black tube or by making the background level at the datum. Open the laserhead shutter.

10. Rotate the analyzer angle \(\phi\), and find out the maximum and minimum intensity. Set the analyzer at angle of 45° approximately w.r.t polarizer.
Note down the value of voltage at the oscilloscope and divide it by 1 MΩ, the input impedance of oscilloscope, to get \( i_{dc} \).

Next, you will determine \( i_{dc} \), the (rms value of the) ac component of the photocurrent (\( \propto \) intensity), using the lock-in amplifier.

11. Activate the magnetic field. Provide the reference signal from the oscillator to the lock-in amplifier. Select the reference frequency mode, \( f \) Hz in the reference channel of lock-in amplifier. Select the current input mode \( I \), from the input section. Make the offset equal to zero. Connect the photodiode to the input BNC connector. Make sure that no error indication (unlock or overload) occurs. Turn on the band pass and line frequency filter. The effect of each key press on the lock-in amplifier is indicated by a nearby LED. Select a suitable sensitivity (usually 200 nA) and time constant (3 ms) for the pre filter. Check that no offset is introduced by the lock-in amplifier at the selected sensitivity scale.

12. Adjust the phase located in reference section to make the output equal to zero. Then introduce a phase shift of 90°, bringing the reference and input signals in-phase.

13. Rotate the analyzer angle \( \phi \) in steps of 10° and tabulate \( i_{ac} \) for any fixed value of the magnetic field. You will observe that the maximum rotation occurs when analyzer is at an angle of approximately 45° relative to polarizer.

**Q 18.** What does the reading on the lock-in amplifier physically represent?

14. Fix the analyzer at 45° relative to the polarizer. Increase the magnetic field, from an initial value of 10 Gauss, in steps of 5 or 10 Gauss by increasing the current. The transverse probe of Gaussmeter can be fixed to observe the magnetic field in the center of Helmholtz coil.

15. Tabulate the values for \( i_{ac} \) for each value of current (and hence the corresponding magnetic field) passing through the coil.

**Q 19.** Use the above results to calculate the Verdet constant of your sample?

**Q 20.** Clearly quantify your uncertainties. What are the major sources of error?

**Q 21.** Can you measure \( i_{dc} \) with the help of the lock-in amplifier?
16. Remove the HeNe laser from setup, place AIGaN diode laser. Turn the diode laser on, its warm up time is 15 minutes and its output is linearly polarized, therefore, remove the polarizer A, use analyzer B only and repeat all the steps.

Applications of Faraday rotation

Optical isolators

An optical isolator acts as a photon valve, passing radiation in one direction and blocking in the other. An isolator is shown in Figure 6.15. Polarizer A is used to make the beam horizontally polarized which is then passed through a 45° Faraday rotator C, followed by another linear polarizer (analyzer, B) at 45° relative to A. If any of the light is reflected or backscattered from analyzer, it undergoes an extra rotation of 45° by C and thus is blocked by A. In a LASER, if any of the emitted light returns into the active medium through an unwanted reflection, it can generate instabilities in the emission. Optical isolators are used to prevent the unwanted reflection in lasers [7].

![Figure 6.15: Optical isolator, backscattered radiation undergoes an additional 45° rotation by C, thus is blocked by polarizer A.](image)

Domain Observation

Light will have different characteristics after reflection or transmission by regions having different orientations of magnetic moments. Let a sample be made up of three domains, the magnetization of each domain is shown in Figure 6.16. Plane polarized light of wavelength λ, passing through domain A, is rotated through some angle θ₁, while interacting with C, is rotated −θ₁. If the analyzer is at −θ₁, A may be dark, C bright and D of intermediate shade. For analyzer at 90°, D will be dark, A and C will be equally bright. For analyzer set at θ₁, A will be bright,
C dark and D of intermediate shade, i.e., polarization direction may be turned one way or the other, depending on magnetization, thus resulting in different intensities, it is therefore possible to image magnetic domains [7].

![Diagram of optical circulator](image)

**Figure 6.16:** The domain imaging through magneto optic rotation.

**Circulator**

Optical circulators are used in fiber optics, to separate light traveling in opposite directions. Figure 6.17 shows one such circulator. It is made up of two Foster Seely Prisms and a 45° rotator placed between the prisms. In these prisms the rejected polarized light is internally reflected, so that it exits perpendicular to the axis of prism. Horizontally polarized light entering along a passes straight through the prism, is rotated to 45° by the rotator using Faraday rotation and emerges from second prism at b. However, any light reflected back to the circulator entering at b with polarization azimuth 45° undergoes a 45° rotation through the rod, thus polarized at 90° and exits from port c. Similarly, light entering at c emerges at d and entering at d exits at a [7]. A circulator has at least three ports. The light beam if entering from one port after passing through circulator exits from the second. Another light beam entering from the second port or light reflected from second port exits from third port and so on.
6.5 **(OPTIONAL) Measurement of the the Verdet constant using higher harmonic components**

The light rotated by the Faraday medium incident on the photodetector from analyzer, contains fundamental as well as components at higher frequencies. The rms values $u_1$ and $u_2$, at $f$ and $2f$, respectively of these current components are measured, where $f$ is the frequency of ac signal passing through Helmholtz coil. The ratios $u_1/U_0$ and $u_2/U_0$ can also be used to determine the Verdet constant, where, $U_0$ is the steady output from the photodiode under zero magnetic field and analyzer set for maximum transmittance [9]. The power transmitted through a Faraday rotator is,

$$I = \frac{A^2}{2} [1 + \cos 2(\phi - \theta)]$$

(6.55)

$$= \frac{A^2}{2} [1 + \cos 2(\phi - \theta_0 \cos(\Omega t))]$$

$$= \frac{A^2}{2} [1 + \cos 2\phi \cos(2\theta_0 \cos(\Omega t)) + \sin 2\phi \sin(2\theta_0 \cos(\Omega t))]$$

Using the Jacobi-Anger expansion, we obtain[10],

$$\cos(2\theta_0 \cos(\Omega t)) = J_0(2\theta_0) + 2 \sum_{m=1}^{\infty} (-1)^m J_2m(2\theta_0) \cos(2m\Omega t)$$

$$\sin(2\theta_0 \cos(\Omega t)) = 2 \sum_{m=1}^{\infty} (-1)^m J_{2m+1}(2\theta_0) \cos((2m+1)\Omega t)$$

where the Bessel function is,

$$J_\alpha(x) = \sum_{q=0}^{\infty} \frac{(-1)^q}{q! \Gamma(q + \alpha + 1)} \left(\frac{x}{2}\right)^{2q+\alpha}$$

and $\Gamma$ is the factorial function, given by."
Therefore, Eq (6.56) becomes,

\[
I = \frac{A_o^2}{2} \left[ 1 + \cos(2\phi) \left( J_0(\theta_o) + 2 \sum_{m=1}^{\infty} (-1)^m J_{2m}(\theta_o) \cos(2m\Omega t) \right) + \sin(2\phi) \left( 2 \sum_{m=0}^{\infty} (-1)^m J_{2m+1}(\theta_o) \cos(2m+1)\Omega t \right) \right].
\]  \hspace{1cm} (6.56)

Let the amplitude of coefficient of the terms containing $\Omega t$ and $2\Omega t$ be represented by $s_1$ and $s_2$ respectively. Then,

\[
s_1 = \frac{A_o^2}{2} 2(-1)^0 J_0(\theta_o)|\sin(2\phi)| \]
\[
= A_o^2 \sum_{q=0}^{\infty} \frac{(-1)^q}{q!\gamma(q+1+1)} \left[ \frac{2\theta_o}{2} \right]^{2q+1} |\sin(2\phi)|
\]
\[
= A_o^2 \left[ \frac{(-1)^0}{0!\gamma(2)} \theta_o + \frac{-1}{1!\gamma(3)} \theta_o^3 + \frac{(-1)^2}{2!\gamma(4)} \theta_o^5 + ... \right] |\sin(2\phi)|
\]
\[
= A_o^2 \left[ \frac{1}{1!} \theta_o^2 + \frac{-1}{2!\gamma(3)} \theta_o^4 + \frac{1}{2!\gamma(5)} \theta_o^6 + ... \right] |\sin(2\phi)|
\]
\[
= A_o^2 \theta_o \left[ 1 - \frac{1}{2} \theta_o^2 + \frac{1}{12} \theta_o^4 + ... \right] |\sin(2\phi)|
\]  \hspace{1cm} (6.57)

\[
s_2 = \frac{A_o^2}{2} 2\theta_o(2\theta_o)|\cos(2\phi)|
\]
\[
= A_o^2 \sum_{q=0}^{\infty} \frac{(-1)^q}{q!\gamma(q+2+1)} \left[ \frac{2\theta_o}{2} \right]^{2q+2} |\cos(2\phi)|
\]
\[
= A_o^2 \left[ \frac{1}{\gamma(3)} \theta_o^2 + \frac{-1}{\gamma(4)} \theta_o^4 + \frac{(-1)^2}{2!\gamma(5)} \theta_o^6 + ... \right] |\cos(2\phi)|
\]
\[
= A_o^2 \left[ \frac{1}{2!\gamma(3)} \theta_o^2 - \frac{1}{3!\gamma(5)} \theta_o^4 + \frac{1}{2!4!\gamma(7)} \theta_o^6 + ... \right] |\cos(2\phi)|
\]
\[
= A_o^2 \theta_o \left[ 1 - \frac{1}{3} \theta_o^2 + \frac{1}{24} \theta_o^4 + ... \right] |\cos(2\phi)|
\]  \hspace{1cm} (6.58)

Since,

\[
\theta_o = VB_o d
\]  \hspace{1cm} (6.60)

Substituting equation (6.60) in (6.58)

\[
s_1 = A_o^2 VB_o d \left[ 1 + \frac{-1}{2} (VB_o d)^2 + \frac{1}{12} (VB_o d)^4 + ... \right] |\sin(2\phi)|
\]
\[
= U_o VB_o d \left[ 1 + \frac{-1}{2} (VB_o d)^2 + \frac{1}{12} (VB_o d)^4 + ... \right] |\sin(2\phi)|.
\]  \hspace{1cm} (6.61)

where, $U_o$ is the steady power on photodetector when polarizers are set for maximum transmittance (in the absence of applied magnetic field).

Substituting equation (6.60) into (6.59), we obtain,

\[
s_2 = \frac{A_o^2}{2} (VB_o d)^2 \left[ 1 - \frac{1}{3} (VB_o d)^2 + \frac{1}{24} (VB_o d)^4 + ... \right] |\cos(2\phi)|
\]
\[
= \frac{U_o}{2} (VB_o d)^2 \left[ 1 - \frac{1}{3} (VB_o d)^2 + \frac{1}{24} (VB_o d)^4 + ... \right] |\cos(2\phi)|.  \hspace{1cm} (6.62)
\]
The $f$ and $2f$ components are determined through lock-in amplifier which displays rms values, so from equation (6.61), the rms value of the first harmonic component of output current (ignoring higher order terms) is,

$$u_1 \approx \frac{U_o V B_o d}{\sqrt{2}} |\sin(2\phi)| = U_o V B d |\sin(2\phi)| \quad (6.63)$$

where $B = B_o/\sqrt{2}$. $B$ represents the rms value of the field measured by the Gaussmeter. Similarly, from (6.62) the rms value of the second harmonic component of output current is,

$$u_2 \approx \frac{U_o}{2\sqrt{2}} (V B_o d)^2 |\cos(2\phi)| = \frac{U_o}{\sqrt{2}} (V B d)^2 |\cos(2\phi)|. \quad (6.65)$$

Both equations (6.63) and (6.65) can be used to determine Verdet constant.

In short, we have three three different means of measuring the Farday rotation,

**Method 1.** The gradient of the plot of $u_1$ or $i_{dc}$ against $B$ for $\phi = 45^\circ$ results in the Verdet constant. This is, in fact, the method you have used in previous section. Since, $U_o = 2i_{dc}$ and $u_1 = i_{dc}$. Equation (6.63) is actually Eq. (6.43) in disguise.

**Method 2.** Determine the gradient of the least squares-fit line to a plot of $u_1/B$ against $|\sin 2\phi|$ for fixed $U_o$. Equate the gradient to $V B d U_o$ and find the Verdet constant [9].

**Method 3.** Determine the gradient of a plot of $u_2$ against $B^2$ when $\phi = 90^\circ$. equate this to $V^2 d^2 U_o/\sqrt{2}$ and find the Verdet constant.

**Q 22.** Find the Verdet constant for TGG at 405 nm using methods 2 and 3.
Chapter 7

Studying Phase Transitions
with a Strain Gage

Muhammad Wasif and Sabieh Anwar

Many physical substances undergo phase transitions when subject to changes in environmental parameters. The transition of ice into water and water into steam are familiar examples. Similarly, chromium undergoes a phase transition at a specific temperature, called the Neél temperature changing its magnetic order from the antiferromagnet to the paramagnet state. This magnetic reordering is accompanied by a change in volume. In other words, the phase transition is a magnetoelastic phenomenon. In this experiment, we study the phase transition of chromium by directly measuring the volume change. Several important thermodynamical concepts will also be navigated.

KEYWORDS

Antiferromagnetism · Paramagnetism · Neél Temperature · Strain Gage · Latent Heat · Gibbs Function · Wheatstone Bridge · Instrumentation Amplifier ·

APPROXIMATE PERFORMANCE TIME 1 week.
7.1 Objectives

The objectives of the present experiment are to,

1. understand the meaning of phase transitions and the order of transitions,

2. how change in volume occurs as the temperature changes and especially at the phase transition,

3. understand the relationship between the paramagnetism and antiferromagnetism, magnetism and thermodynamics, and thermodynamics and elastic properties of materials,

4. learn the use of the strain gage,

5. understand the purpose and significance of the instrumentation amplifier and the bridge circuitry which are general-purpose measured techniques.
Bibliography


[4] Thermal and transport properties of chromium, Ch. 6


7.2 Introduction

Chromium acts as an antiferromagnet below the Néel temperature but at the Néel temperature \( T_N \approx 38^\circ \text{C} \), it undergoes a phase transition and becomes paramagnetic. This phase transition occurs due to the change in the orientation of magnetic dipoles as shown in Fig 1. Thermodynamically this magnetic transition is of first order. We can easily detect this transition by measuring the volume change \( \Delta V \) in chromium.

In antiferromagnetic materials such as chromium, the interaction between the magnetic moments tends to align adjacent moments antiparallel to each other as illustrated in Fig. 1. Although one set of magnetic ions is spontaneously magnetized below some critical temperature (called the Néel Temperature, \( T_N \)), the second set is spontaneously magnetized by an equal amount in the opposite direction. As a result, antiferromagnets have no net spontaneous magnetization. Excellent resources on magnetism in materials are the references [1] and the more detailed [2].

Antiferromagnets are being widely used in many applications such as GMR sensors, spin valves, MRAM’s and computer hard disks where antiferromagnets are coupled to ferromagnets through the exchange interaction. Antiferromagnetic materials occur commonly among transition metal compounds, common examples include heavy-fermion superconductor URu2Si2, alloys such as iron manganese (FeMn), oxides such as nickel oxide (NiO) and the most popular example, manganese oxide (MnO).

Q.1: Why do we prefer measuring the volume change at the Néel transition instead of directly detecting the magnetic reordering?

First Order Phase Transition

In thermodynamics [3], the change of state such as ice melting to water or water vaporizing into steam are referred to as transitions of phase. This transition in phase accompanies change in entropy \( S \) and volume \( V \) while the pressure \( P \) and temperature \( T \) remain constant. The first order phase transition can be defined
Figure 7.1: Phase transition of chromium from (a) antiferromagnetic to (b) paramagnetic phase at $T_N$. In the antiferromagnetic lattice, neighboring spins are anti-aligned. The antiferromagnetic state can be represented by two ferromagnetic sublattices $A$ and $B$ which are aligned antiparallel to each other. These sublattices are represented by filled and empty circles in (a).

by considering the Gibb's function $G(P, T)$. Note that the natural variables for $G$ are $P$ and $T$ and is defined through the Legendre transform,

$$
G = H - TS. 
$$

If the process is reversible and infinitesimal, we have the characteristic equation,

$$
dG = VdP - SdT. 
$$

**Q 2.** Derive Equation (2) from (1) and the first law of thermodynamics.

Since $dG$ is an exact differential and the variables $P$ and $T$ are independent, one can write the molar entropy and volume as differentials of the Gibb's energy,

$$
S = -\left(\frac{\partial G}{\partial T}\right)_P 
\text{ and } 
$$

$$
V = \left(\frac{\partial G}{\partial P}\right)_T. 
$$

Considering these relationships, we say that the phase transition is of first order only if it satisfies the following equivalent requirements.

1. The molar entropy and molar volume change at the phase transition.

2. The first-order derivatives of the Gibb's function with respect to temperature and pressure change discontinuously. This predicted behavior is shown in Fig. 2.

**Q 3.** What is a second order phase transition?
Figure 7.2: Characteristics of a first-order phase transition. There are discontinuous changes in (a) molar entropy and (b) molar volume whereas the (c) Gibbs function is single valued with a discontinuous slope.

**Q 4.** Enumerate common first and second order phase transitions.

**Q 5.** What is the physical meaning and significance of Gibbs free energy?

Volume is a directly observable parameter, whereas entropy is impossible to measure directly. In the current experiment, we will monitor the change in volume that occurs in Cr as its temperature is varied expecting a discontinuous change at the Neél temperature.

**Clausius-Clapeyron Equation**

Phase transitions are generally understood with the help of phase diagrams that constitute powerful tools in thermodynamics and materials science. A phase diagram characterizes a discontinuous phase transition between two phases on a pressure–temperature ($P$-$T$) graph. Consider the example of phase transition of ice melting into water and then water into vapors, this transition is depicted in the $P$-$T$ diagram of Figure 3. The line separating two phases is known as a coexistence curve. Along the coexistence curve, $dG = 0$ or $G =$ constant. The Clausius-Clapeyron equation [3] gives the slope of this coexisting curve, $dP/dT$ and is applicable to first-order phase transitions, for which $dP = dT = 0$ along the phase transition. The equation is,

$$
\frac{dP}{dT} = \frac{\Delta H}{T \Delta V} = \frac{L}{T \Delta V}.
$$

(7.5)

where, $L$ is the latent heat involved in the isobaric phase transition, $\Delta V$ is the change in volume during the phase transition, and $\Delta H = L$ is the difference in enthalpies of the final and initial phases.
7.3 Overview of the Experiment

A thermocouple and a strain gage have been attached to a piece of polycrystalline chromium metal as shown in Figure 4. The setup is placed in a sand bath for temperature stability and uniformity. The bath is then heated slowly on a hot plate and the output voltage of the instrumentation amplifier connected to a bridge circuit incorporating a strain gage, is plotted against temperature.
Figure 7.4: Experimental Setup.

**Volume Measurement with the strain gage**

When chromium undergoes a phase transition at Néel temperature, not only is its magnetic order transformed, there is also a change in its volume. We can measure this change with the help of a strain gage and some useful electronics. The thermally induced volume expansivity is characterized by the variable,

\[ \beta = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_p, \]  

(7.6)

and using the characteristic equation (7.2) as well as the Maxwell’s relations, we can derive,

\[ T \, dS = C_p dT - V \beta dP, \]  

(7.7)

\( C_p = T(\partial S/\partial T)_p \) being the heat capacity at constant pressure. For a phase transition, both \( dT \) and \( dP \) are zero, but since the heat transferred during the phase transition \( T \, dS \) is finite, we have \( C_p \to \infty \) and \( \beta \to \infty \), indicating a discontinuous change in the volume at the phase transition.

The strain gage is directly glued to the metal with the help of specialized adhesives and bonding pads [5, 6]. The fractional change in the resistance of strain...
gage is related to the fractional volume change in the polycrystalline chromium. The volume expansivity $\beta$ and the linear expansivity (of any material) are related through the simple relationship:

$$\beta = \frac{3 \Delta l}{\Delta T l}$$

(7.8)

**Q 9.** Verify the relationship (7.8), which is only true for a cubic single crystal or polycrystalline material with randomly oriented crystallites.

**Q 10.** Derive equation (7.7).

Figure 7.5: Linear strain gage, with bonding pads. Connective wires are soldered to the bonding pads instead of being directly attached to the gage wire.

The strain gage, for example the one shown in Figure 5, is a device used to measure the strain $\Delta l/l$ of an object to which it is glued. The output resistance of the strain gage is a function of strain and can be interrelated with the help of a gage factor $F$.

$$F = \frac{\Delta R}{R \Delta l / l}.$$ 

(7.9)

In this experiment we are using the LY11 strain gage from Omega Engineering [6]. Table 1 provides some important characteristics of LY11 supplied by the manufacturer.

<table>
<thead>
<tr>
<th>Gage Factor</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Temperature compensation</td>
<td>Steel</td>
</tr>
<tr>
<td>Nominal resistance at room temperature</td>
<td>120 $\Omega \pm 1%$</td>
</tr>
<tr>
<td>Active gage length at room temperature</td>
<td>1.50 $\pm 0.059$ mm</td>
</tr>
<tr>
<td>Max voltage drop allowed</td>
<td>2.5 $V_{rms}$</td>
</tr>
</tbody>
</table>

**Table 1.** Characteristics of Omega’s LY11 strain gage. See the website [6] for more details.


Temperature measurement with the thermocouple

The thermocouple is a transducer that converts temperature into voltage. By now, you will be familiar with its operation. In the present experiment we using a J-type thermocouple. The thermocouple has been pre-calibrated and is directly connected to the Data Acquisition (DAQ) board which has a built-in thermistor for cold junction compensation. Table 2 provides some important characteristics of the J-type thermocouple. You are recommended to consult the strain gauge and thermocouple data sheets which also are available on the Physlab website.

<table>
<thead>
<tr>
<th>Temperature Range</th>
<th>-40 to +750°C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Combination of alloys</td>
<td>chromel and constantan</td>
</tr>
<tr>
<td>Sensitivity</td>
<td>Approx. 55μV/°C</td>
</tr>
</tbody>
</table>

*Table 2. Some relevant characteristics of the J-Type thermocouple.*

Coefficient of Thermal Expansion

Since the strain gage is made of steel and has its own thermal expansion, so when the strain gage is bonded to the object (in our case chromium), the strain gage bears thermally induced apparent strain per degree change in temperature. The thermally induced strain per unit temperature change is given by the equation,

\[
\frac{\Delta l}{l} = \frac{\alpha}{F} + |\beta_s - \beta_c|.
\]

(7.10)

where, \( \alpha \) is the temperature coefficient of resistance of the strain gage and for steel its value is 6.25x10^{-10} K^{-1} [13], \( F \) is the gage factor of the strain gage, \( \beta_s \) is the coefficient of thermal expansion of steel strain gage, and \( \beta_c \) the coefficient of thermal expansion of chromium. The thermal expansivity of steel can be looked up from reference tables such as [10] or the CRC Handbook.

Some specialized electronics is required to measure the strain with a strain gage. In this experiment we use a Wheatstone bridge and an instrumentation amplifier to directly measure the change in resistance induced by the change in volume.
Wheatstone Bridge

The Wheatstone bridge is normally used to measure an unknown resistance when placed in the bridge configuration, the arrangement of a typical bridge is shown in Figure 6.

![Wheatstone Bridge Diagram](image)

Figure 7.6: Wheatstone bridge configuration for measuring $R_g$.

In our circuit $R = 120 \Omega$, $R_g$ is the resistance of the strain gage and $R_{adj}$ is the variable resistance used to balance the bridge. Before heating the metal, we ensure that $R_{adj}$ is adjusted such that the voltage reading at the output of the bridge is zero ($V_g = 0$). We can then use the bridge equation to evaluate the change in resistance $R_g$,

$$V_g = \left( \frac{R_g}{R + R_g} - \frac{1}{2} \right) V_s$$

where, $V_s \approx 1.5 \text{ V}$, is the supply voltage and $V_g$ is the output voltage.

**Q 21.** Derive equation (8) from first principles of circuit analysis.

Instrumentation Amplifier

An instrumentation amplifier is used when high and stable gain is required, especially in the amplification of weak signals, say, in the order of microvolts. Ordinary amplifiers have very low noise rejection ability and hence when they are fed in with very small input signals, the output signal gets distorted. For the same reason the instrumentation amplifier is the most preferred device in applications involving weak signal measurements such as ECG and EEG, strain gage systems to detect
cracks in masonry as well as precise resistance thermometers.

In the current experiment, we use the AD624 instrumentation amplifier [9]. The block diagram of AD624 is shown in figure 7. It can be noted that the instrumentation amplifier is a combination of ordinary amplifiers with some precise matching resistances. **We use AD624 with the gain value configured at 500.**

![Block Diagram of the Instrumentation Amplifier AD624](image)

**Figure 7.7: Block Diagram of the instrumentation amplifier AD624.**

### 7.4 Experimental Procedure

1. Open and run the labview file titled **Cr-phase-study.vi**. The programme will start acquiring and saving data from the thermocouple and the strain gage. The output graphs and other details on the front end are self-explanatory.

2. Observe the temperature reading. It should be around room temperature.

3. Now turn on the hotplate and set the knob to its minimum value ensuring extremely slow heating of the sand bath.

4. Again observe the temperature reading which is continuously rising as the bath is heated.

5. Stop the labview file when the temperature has reached around 50°C.

6. Plot $V$ (instrumentation amplifier output voltage) against temperature $T$ using the Matlab file **Exp-graphs.m**.
7.5 Experimental Objectives

1. By identifying the transition in the V· T plot, estimate the Néel Temperature $T_N$.

2. What is the estimated uncertainty or standard error in $T_N$?

3. Note down the dip in the value of $V_g$ at $T_N$.

4. Find the change in resistance ($\Delta R$) of the strain gage with the help of the bridge equation. Note: Before using the bridge equation compensate the value of $V_g$ with the AD624 gain $G$.

5. What is the uncertainty in the value of $\Delta R$?

6. Calculate the value of the latent heat ($L$) per unit volume associated with the phase transition at Néel Temperature. Use the Clausius-Clapeyron relation, with $dT/dP = -1.3 \times 10^{-7}$ K/Pa [4]. The chromium sample used in this experiment has a mass of 8.6 g. Compare your latent heat with values published in [12].

7. How has the uncertainty propagated from $\Delta R$ into $L$? Show the complete steps and calculations.

8. Calculate the change in entropy at the Néel temperature.

9. Determine the values of $\beta_c$ (the coefficient of thermal expansion of chromium) above and below the Néel temperature. What is the uncertainty in the value of $\beta_c$? Do not lose sight of the thermally induced strain of the strain gauge (7.10), that must be accounted for in your calculations.

10. For $T < T_N$, is $\beta_c$ positive or negative? If solid Cr is subject to external pressure, how will the entropy react? Will the Cr atoms become more or less orderly as the pressure is isothermally increased? HINT: Derive the Maxwell relation corresponding to the characteristic function $dG = -SdT + VdP$. 
Chapter 8

Statistical properties of White Noise (Electronics & Signal Processing)

Umer Hassan and Muhammad Sabieh Anwar

The experiment gives you an introduction to noise and its analysis. Our main objective in this experiment is to investigate many statistical properties of white noise. This experiment is divided into sections, such that each section introduces one of the key concepts, and finally this culminates to our final objective.

KEYWORDS

Noise · Autocorrelation · White Noise · Power Spectral density · Probability density function · Standard Deviation · Effective Noise Power density

APPROXIMATE PERFORMANCE TIME 1 week

8.1 Experimental Objectives

The experimental objectives include:

1. understanding the concept of noise
2. understanding correlation and autocorrelation
3. filtering the white noise
4. understanding probability density function, and
5. measuring effective noise power density.
Bibliography


8.2 Filtering White Noise

Objective

This section introduces the basic concepts of white noise, correlation, autocorrelation, and the effect of low pass filtering on white noise. The major impetus of this section comes from reference [1]. The circuit that generates the noise from pseudo-random bit sequences was designed in-house. We also designed the PCB that was fabricated from the hobbyist market. The complete details (including the circuit diagram) are provided on our website whereas the functional block diagram for the experiment is shown in Figure 8.1.
8.2. FILTERING WHITE NOISE

Figure 8.1: Functional block diagram for the low pass filtering of white noise.

**Noise and its types**

Anything that is mostly undesirable or unwanted is referred to as noise. Noise is inevitable in electric circuits. Noise can block, distort, or change the meaning of a message in both human and electronic communication. One form of electronic noise exists in all circuits and devices and is a result of thermal noise, also referred to as Johnson noise. In any electronic circuit, there exist random variations in current or voltage caused by the random movement of the electrons. Pink noise or $1/f$ noise is noise whose graph between the noise power density versus frequency is proportional to the reciprocal of the frequency. It is sometimes also referred as flicker noise. The pink noise spectrum is shown in Figure 8.2(a).

![Figures](image)

Figure 8.2: Power spectral density (noise power per unit frequency) versus frequency for (a) pink noise and (b) white noise.

**White Noise**

This is the type of noise in which all frequency components, ranging from zero frequency (DC) to infinite frequencies, are present. The graph between the noise
power density versus frequency would be a constant line, and this continues to infinite frequencies. Thus it is safe to say that white noise has all frequency components. The white noise spectrum is shown in Figure 8.2(b).

In the present experiment, we will:

1. generate a pseudo-random binary signal (with logic levels +5 V and 0 V),
2. convert it into a symmetric bipolar signal (with logic levels +5 V and −5 V), and
3. finally, low pass filter the signal and observe the autocorrelation of the output. (The autocorrelation will be discussed shortly.)

Correlation between signals

Correlation between two signals is the measure of the similarity or dissimilarity between them. Finding correlation between two signals is extremely important. This concept is widely used for signal processing applications in radar, sonar, and digital communications.

Autocorrelation

Autocorrelation is the correlation of the signal with itself. Informally, it is a measure of how well a signal matches a time-shifted version of itself, as a function of the amount of time shift. The autocorrelation of any signal is computed as follows,

1. Shift the signal in time.
2. Multiply each point of the original signal with the corresponding point in the time shifted signal.
3. Find the sum of all these products.
4. Divide by the total number of points. The answer is the value of the autocorrelation.

A high value of correlation (close to 1) indicates that the time-shifted and original signals are highly correlated. This means that the signal retains some kind of “memory”—the future samples are correlated with the present and past samples.
A small value for the autocorrelation indicates that the signal quickly “forgets” its present values, suggestive of the presence of high frequency components, forcing the signal to change rapidly.

**Autocorrelation of ideal white noise**

White noise is composed of all possible frequency components. The existence of the high frequency components ensures that the values of the noise at any two instances are independent of each other. The noise values are changing rapidly, effectively they represent a random variable. Therefore, the autocorrelation of ideal white noise is zero for all non-zero time shifts. The autocorrelation function for ideal white noise is shown in Figure 13.7 and the concept is sketched out in Figure 8.4.

![Autocorrelation function for ideal white noise.](image)

**Figure 8.3:** Autocorrelation function for ideal white noise.

(a) with no shift

(b) with shift

![Autocorrelation of Ideal White Noise represented as the pointwise multiplication of noise with its time shifted signals. If the noise has a mean of zero, there will be as many positive products as there are negative products. Hence they will all sum up to zero.](image)
Probability Density Function

White noise has many interesting statistical properties in addition to autocorrelation. A histogram is a good approximation to a probability density function (pdf) for a large number of data points. If we empirically sample numerous enough values of a continuous random variable, and make a histogram showing the relative frequencies of output ranges, the histogram will approximate the random variable's probability density.

Procedure

★ Q 1. Download the schematic from http://ravi.lums.edu.pk/physlab/docs/circuit_noise.pdf. Build the circuit on the Bread Board using the provided components.

★ Q 2. Carry out the following experimental procedure to investigate the various statistical properties of white noise.

1. Run the Labview file noise.vi.

2. In the filename option, type in a filename of your choice such as n1.txt.

3. In the front panel window click the Run button (shown by the arrow key).

4. Now observe the following signals.
   (a) At 1, the output from the first multivibrator
   (b) At 2, the output from the second multivibrator
   (c) At 3, the digital noise appearing as a pseudo-random sequence of logic levels
   (d) At 4, the level shifter output, once again a pseudo-random sequence of bipolar logic levels.

5. Stop data acquisition using the Stop button.

6. Design a low pass RC filter with a cut-off frequency of approximately 200 Hz.

7. Connect the output 4 to the filter.

8. Now, run the circuit again, type in your favourite filename (such as n1.txt) and observe the output across the capacitor.
9. On the front panel, click the button **Write 1000 data points**. This will save 1000 points from the filtered output in the file `n1.txt`.

10. After a delay of a few seconds, click the **Write 1000 data points** button again and repeat about ten times, thereby saving $1000 \times 10$ points, all of them successively arranged in the file `n1.txt`.

11. Stop the data acquisition.

12. Start Matlab and load the data saved from Labview using the command,

   \[ \texttt{load('n1.txt');} \]

13. Write the Matlab code to find the autocorrelation of the white noise.

14. Now design low pass filters for the following cut-off frequencies and repeat the above procedure, 10 Hz, 50 Hz, 100 Hz, 500 Hz, 1000 Hz.

**Q 3.** Plot the autocorrelation functions for different cut-off frequencies and compare between these plots.

**Q 4.** How do the autocorrelation functions depend on the cut-off frequencies? Come up with a suitable strategy to plot the relationship between autocorrelation and cut-off frequency. Discuss with your demonstrator.

**Q 5.** What is the advantage of saving $1000 \times 10$ data points for each value of the cut-off frequency instead of a 1000 points?

**Q 6.** Plot the probability density function at the above cut off frequencies. Also calculate the mean and standard deviation as well.
Chapter 9

Michelson Interferometry

Umer Hassan and Muhammad Sabieh Anwar

Michelson Morley’s interferometer together with Einstein’s theory of relativity helped in abandoning the concept of ether in the classical experiment performed in 1887. Since then, interferometers are used for many purposes, e.g., one of the most exciting and grand goals is measuring gravitational waves. The Laser Interferometer Gravitational-wave Observatory (LIGO), is a collaborative project of American universities aimed at detecting waves of gravitational origin. The goal of the current experiment is to familiarize students with the Michelson Interferometer. It introduces optical components like beam splitters, translation and rotational stages, lenses and highlights important concepts in wave interference. Michelson interferometer is used for measuring the laser wavelength and finding the refractive index of glass.

KEYWORDS

Interferometer · Beam Splitters · Optical alignment · refractive index · constructive and destructive interference · optical path length ·

APPROXIMATE PERFORMANCE TIME 1 week.

9.1 List of Equipment

1. HeNe Laser with Mount, HR020, Thor Labs
9.2 Experimental Objectives

The experimental objectives include:

1. working with optical components,
2. optical alignment of the interferometer,
3. Understanding the concept of change in optical path length,
4. measuring the wavelength of the HeNe laser, and
5. finding refractive index of thin glass sheet.
Bibliography


9.3 Introduction

An interferometer exploits the interference of light, a wave property. A fringe pattern results from optical path differences. The Michelson Interferometer is an amplitude-splitting interferometer. Figure 12.2 shows the schematic whereas Figure 9.2 is a photograph of a typical setup you will have to build in the lab. The working of the interferometer is described in detail [2, 3] and you are required to carefully read the background material.
9.4 Optical alignment of the Michelson interferometer

Q 1. Perform the following procedure to align the interferometer.
1. Place the HeNe laser into laser mount which is bolted on the optical bread-
board. Align the laser parallel to the breadboard.

2. Now mount the beam splitter (BS) on the breadboard in front of laser.

3. Place a fixed mirror (M1) in the mirror mount assembly. Align the mirror
such that the reflected beam from BS falls in the middle of it.

4. Now mount the other mirror (M2) on the translation stage which is being
operated by the computer controlled servo motor. Align M2 such that the
transmitted beam from BS falls in middle of it.

5. Turn ON the laser, you shall see the three small dot images. Using the
screws of the mirrors, overlap all three dots. Now the two beams will be
aligned.

6. Place the plano-convex lens of focal length 35 mm (L1) in between laser
and BS.

7. On the opposite side of M1 place another plano-convex lens of focal length
25.4 mm before the target screen to enlarge the image.

8. Carefully adjust M1 and M2 using their screws so that the beams overlap
on the target.

9. Perform alignment until you see concentric circular fringes.

9.5 Measuring wavelength of the laser using Michelson Interferometer

If we move the movable mirror we see the fringes entering or exiting the center of
the fringe pattern, depending upon the direction in which the mirror moves. The
following equation relates the the distance moved by the mirror with the number
of fringes passed by.

\[ \lambda = \frac{2\Delta d}{N} \]  

(9.1)

where, \( \lambda \) is the laser wavelength, \( N \) is number of fringes passed by and \( \Delta d \) is the
distance moved by M2. You must understand the derivation of Equation (9.1).

Q 2. Perform the following procedure to measure the wavelength of the HeNe
laser.
1. Use the computer software to move M2 using the servo controlled motor by a fixed distance of around 10 um. The software Instructions and Servo motor specifications are given in Appendix A.

2. Count the number of fringes that passes by a fixed point.

3. Fringes shall be entering on exiting center depending on the direction in which you move M2.

4. Repeat the same procedure at least 5 times and takes the average to minimize the errors.

5. Use equation (9.1) to measure the laser wavelength. What is your uncertainty in the wavelength?

### 9.6 Measuring refractive index of glass

Glass slide is mounted on the rotational stage and placed in between BS and M1. If we rotate the glass slide we see the fringes entering or exiting the center of the fringe pattern, depending upon the direction in which the rotation takes place. The refractive index is,

\[ n_g = \frac{(2t - N\lambda)(1 - \cos\theta)}{2t(1 - \cos\theta) - N\lambda} \]  \hspace{1cm} (9.2)

where, \( n_g \) is the refractive index of glass, \( \lambda \) is the laser wavelength, \( t \) is the thickness of glass slide, \( N \) is number of fringes passed by and \( \theta \) is the angle rotated in radians.

**Q 3.** Draw a figure showing the path of the light beam as it passes through the glass slide at an arbitrary angle \( \theta \) with respect to the light ray. What is the optical path length difference in this case?

**Q 4.** Derive Equation (9.2).

**Q 5.** Plot \( N \) versus \( \theta \) as elicit by Equation (9.2).

**Q 6.** Perform the following procedure to find the refractive index of glass slide.

1. Find the thickness of the glass slide.

2. Place the glass slide on the rotational stage in between M1 and BS such that it is exactly parallel to M1.
3. Rotate the glass slide by certain number of degrees and count the number of fringes that passes by a fixed point.

4. Fringes shall be entering on exiting center depending on the direction in which you rotate the glass slide.

5. Use equation (9.2) to compute the refractive index of glass.

6. Repeat the same procedure at least 5 times and take the average result to minimize the error. Remember that the relationship (9.2) between \( N \) and \( \theta \) is nonlinear.

7. Calculate the uncertainty in \( n_y \) arising from the uncertainties in \( N \) and \( \theta \).

Q 7. Does your experimental results verify the relationship?

9.7 Appendix A

Software Instructions

1. Open "APT Configuration Utility", and select the following Motor and Stage options.
   Motor 83823465, Channel 1 and Stage Z825.

2. After configuration, run "APT User Utility".

3. Select the "Motor driver Settings" tab and enter the following parameters.
   (a) In Moves-Velocity Profile,
      Max. Vel. = 0.0003 mm/sec,
      Acc./Dev. = 0.03 mm/sec/sec.

   (b) Step distance = 0.001 mm.

4. After completing the settings, right click on the panel window and select "Graph view". On right side of the screen you can see the Channel position (current position) and cursor position (desired position). Adjust the cursor position such that the difference in between both positions is around \( \Delta d \).

5. Click the moves and stop button to operate the servo motor.
DC Servo Motor Specifications

Travel range = 25 mm
Lead Screw Pitch = 1 mm
Resolution = 29 nm
Chapter 10

Band Structure and Electrical Conductivity in Semiconductors

Amrozia Shaheen, Wasif Zia and Muhammad Sabieh Anwar

Semiconductors are one of the technologically most important class of materials. According to the band theory of solids, which is an outcome of quantum mechanics, semiconductors possess a band gap, i.e., there is a range of forbidden energy values for the electrons and holes. In this experiment, we will calculate the energy band gap in the intrinsic region and the temperature dependence of the majority carrier mobility in the extrinsic region.

KEYWORDS

Semiconductor · intrinsic conduction · extrinsic conduction · energy band gap · conduction band · valence band · conductivity · resistivity · mobility · unijunction transistor · temperature control · low temperature physics

Approximate Performance Time 2 weeks.
10.1 Objectives

In this experiment, we will,

1. understand how conductivity in semiconductors depends on carrier concentration and mobility, and how these depend on temperature,

2. distinguish between intrinsic and extrinsic temperature regimes and identify the applicable temperature range from an examination of measured data,

3. appreciate and utilize the advantages of the four-probe resistance measurement technique,

4. calculate the energy band gap for doped Si and pure Ge,

5. calculate the temperature dependent coefficient $\alpha$ of the majority carriers,

6. through experimental realizations, appreciate a physical understanding of the band gap structure of semiconductors.
Bibliography


10.2 Theoretical introduction

Semiconductors

The available energies for electrons help us to differentiate between insulators, conductors and semiconductors. In free atoms, discrete energy levels are present,
but in solid materials (such as insulators, semiconductors and conductors) the available energy states are so close to one another that they form bands. The band gap is an energy range where no electronic states are present. In insula-

![Diagram of电子能级结构]

Figure 10.1: Simplified diagram of the electronic band structure of insulators, semiconductors and metals. The position of the Fermi level is when the sample is at absolute zero temperature (0 K).

tors, the valence band is separated from the conduction band by a large gap, in good conductors such as metals the valence band overlaps the conduction band, whereas in semiconductors there is a small gap between the valence and conduction bands, small enough allowing thermal excitation of electrons from the valence to conduction band. The overall picture is shown in Figure (10.1).

The Fermi level is an important consequence of band theory, the highest occupied quantum state of electrons at absolute zero temperature. The position of the Fermi level relative to the conduction band is an important parameter that contributes to determine the electrical properties of a particular material. The position of the Fermi level position is also indicated in Figure (10.1).

For a semiconductor, the electrical resistivity lies between a conductor and an insulator, i.e., in the range of $10^3$ Siemens/cm to $10^{-8}$ S/cm. An externally applied electrical field may change the semiconductor’s resistivity. In conductors, current is carried by electrons, whereas in semiconductors, current is carried by the flow of electrons or positively charged holes.

Q 1. Explain (or sketch) the temperature dependence of resistance for metals and semiconductors. Why does the resistance of a semiconductor decrease with increasing temperature?
Intrinsic and extrinsic semiconductors

An intrinsic semiconductor is a pure semiconductor having no impurities. In an intrinsic semiconductor, the numbers of excited electrons and holes are equal, i.e., \( n = p \) as shown in Figure (10.2a). A semiconductor in which doping has been introduced, thus changing the relative number and type of free charge carriers, is called an extrinsic semiconductor.

![Energy band diagrams](image)

Figure 10.2: Energy band diagrams for (a) intrinsic, (b) n-type, and (c) p-type semiconductors. \( E_F \) is the Fermi energy level, and the letters \( i, n, p \) indicate intrinsic, n and p-type materials. \( E_C \) and \( E_V \) are the edges of the conduction and valence bands.

An extrinsic semiconductor, in which conduction electrons are the majority carriers is an n-type semiconductor and its band diagram is illustrated in Figure (10.2b), one in which the holes are the majority charge carriers is a p-type semiconductor and is indicated in Figure (10.2c). In extrinsic semiconductors, the dopant concentration \( N_d \) is much larger than the thermally generated electron-hole pairs \( n_i \) and is temperature independent at room temperature.

Q 2. Why is doping introduced in semiconductors? How does it effect the conductivity of a semiconductor?

The ubiquitous role of semiconductor devices

Semiconductor devices are the foundation of the electronic industry. Most of these devices can be constructed from a set of building blocks. The first building block is the metal-semiconductor interface as shown in Figure (10.3a). This interface can be used as a rectifying contact, i.e., the device allows current in one direction as in ohmic contact. By using the rectifying contact as a gate, we
can form a MESFET (metal-semiconductor field-effect transistor), an important microwave device.

Figure 10.3: Basic device building blocks of (a) metal-semiconductor interface, (b) p-n junction, (c) heterojunction interface, (d) metal-oxide-semiconductor structure.

The second building block is the p-n junction, a junction of p-type and n-type materials indicated in Figure (10.3b). The p-n junction is the key compound for numerous semiconductor devices. By combining two p-n junctions, we can form the p-n-p bipolar transistor, and combining three p-n junctions to form a p-n-p-p structure, a switching device called a thyristor can be formed.

The third important building block is the heterojunction interface depicted in Figure (10.3c). It is formed between two dissimilar semiconductors, for example gallium arsenide (GaAs) and aluminium arsenide (AlAs) and is used in band gap engineering. Band gap engineering is a useful technique to design new semiconductor devices and materials. Heterojunctions and molecular beam epitaxy (MBE) are the most important techniques in which required band diagrams are devised by continuous band-gap variations. A new generation of devices, ranging from solid-state photomultipliers to resonant tunneling transistors and spin polarized electron sources, is the result of this technique.

The fourth building block is the metal-oxide-semiconductor (MOS) structure. It is a combination of a metal-oxide and an oxide-semiconductor interface indicated as in Figure (10.3d). The MOS structure used as a gate and the two semiconductor-metal oxide junctions are the source and drain; the result is the MOSFET (MOS field-effect transistor). The MOSFET is the most important component of modern integrated circuits, enabling the integration of millions of devices per chip.
Conduction in intrinsic semiconductors

The process in which thermally or optically excited electrons contribute to the conduction is called intrinsic semiconduction. In the absence of photonic excitation, intrinsic semiconduction takes place at temperatures above 0 K as sufficient thermal agitation is required to transfer electrons from the valence band to the conduction band [3].

The total electrical conductivity is the sum of the conductivities of the valence and conduction band carriers, which are holes and electrons, respectively. It can be expressed as

\[ \sigma = n_e q_e \mu_e + n_h q_h \mu_h. \]  \hspace{1cm} (10.1)

where \( n_e, q_e, \) and \( \mu_e \) are the electron's concentration, charge and mobility, and \( n_h, q_h, \) and \( \mu_h \) are the hole's concentration, charge and mobility, respectively.

![Diagram](image)

Figure 10.4: Band gap structure of an intrinsic semiconductor. (a) Schematic band diagram, (b) density of states \( g(E) \), (c) Fermi distribution function \( f(E) \), (d) carrier concentration \( n_e(E) \) and \( n_h(E) \). \( E_c, E_v \) and \( E_f \) represent the conduction band energy, valence band energy and Fermi energy level, respectively.

The mobility is a quantity that directly relates the drift velocity \( v_d \) of electrons to the applied electric field \( E \) across the material, i.e.,

\[ v_d = \mu E. \]  \hspace{1cm} (10.2)
In the intrinsic region the number of electrons is equal to the number of holes, so Equation (10.1) implies that,

\[ \sigma = n_e q_e (\mu_e + \mu_h). \]  

(10.3)

The electron density (electrons/volume) in the conduction band is obtained by integrating \( g(E) f(E) dE \) (density of states x probability of occupancy of states) from the bottom to top of the conduction band,

\[ n_e = \int_{E_i}^{\infty} g(E) f(E) dE. \]  

(10.4)

There are two important quantities introduced in the above expression: \( g(E) \) is the number of states per unit energy per unit volume known as the density of states. The density of states in the conduction band can be derived from first principle and is given by,

\[ g(E) = \frac{(\sqrt{2})}{\pi^2 h^3} \left( E - E_c \right)^{1/2}. \]  

(10.5)

The function \( f(E) \) is the probability of an electronic state of energy \( E \) being occupied by an electron, and is given by the Fermi-Dirac distribution function,

\[ f(E) = \frac{1}{1 + \exp \left( \frac{E - E_f}{k_B T} \right)}. \]  

(10.6)

The profiles of \( g(E) \) and \( f(E) \) are depicted in Figure (10.4). If we suppose that \( E - E_f \gg k_B T \), then Equation (10.6) can be approximated as,

\[ f(E) \approx \exp \left( - \frac{E - E_f}{k_B T} \right). \]  

(10.7)

Thus, we can replace the Fermi-Dirac distribution by the Boltzmann distribution under the assumption that the number of electrons in the conduction band is far less than the number of available states in this band (\( E - E_f \) is large as compared to \( k_B T \)).

The number of mobile charge carriers (i.e., \( n_e \) in the conduction band and \( n_h \) in the valence band) can be obtained by performing the integration in Equation (10.4), and is given by,

\[ n_e = N_c \exp \left( -\frac{(E_c - E_f)}{k_B T} \right). \]  

(10.8)

and

\[ n_h = N_v \exp \left( -\frac{(E_f - E_v)}{k_B T} \right). \]  

(10.9)

where

\[ N_c = 2 \left( \frac{m^* k_B T}{2 \pi \hbar^2} \right)^{3/2}. \]  

(10.10)
\[ N_v = 2 \left( \frac{m_v^* k_B T}{2 \pi \hbar^2} \right)^{3/2}. \tag{10.11} \]

\( N_c \) and \( N_v \) are the effective density of states for the edges of conduction and valence bands, respectively [1].

**Q 3.** Derive the expressions (10.10) and (10.11) for the effective density of states for the conduction band, \( N_c \), and for the valence band, \( N_v \).

The terms \( m_e^* \) and \( m_h^* \) are the effective masses of electrons and holes respectively, \( k_B \) is Boltzmann’s constant, \( T \) is the absolute temperature, and \( \hbar \) is Planck’s constant.

**Q 4.** What do you understand by the term ‘effective mass’ of an electron? How is it different from the conventional electron mass?

In an intrinsic semiconductor, the number of electrons is equal to the number of holes, so the charge carrier concentration is given by,

\[ n_i = \sqrt{n_e n_h} = \left( N_c N_v \right)^{1/2} \exp \left( \frac{-E_g}{2 k_B T} \right). \tag{10.12} \]

where, \( E_g = E_c - E_v \) is the energy band gap. The term \( (N_c N_v)^{1/2} \) in Equation (10.12) depends on the band structure of the semiconductor. It will be shown later that for intrinsic behavior, \( n_i \) varies as some power of \( T \), so Equation (10.12) can be written as,

\[ n_i = C T^{3/2} \exp \left( \frac{-E_g}{2 k_B T} \right). \tag{10.13} \]

where, \( C \) is some constant. Substituting the expression (10.13) into (10.3) yield the following expression for the intrinsic conductivity,

\[ \sigma = C T^{3/2} q_e \left( \mu_e + \mu_h \right) \exp \left( \frac{-E_g}{2 k_B T} \right). \tag{10.14} \]

Equation (10.14) shows that the electrical conductivity of intrinsic semiconductors decreases exponentially with increasing temperature.

**Q 5.** Derive Equation (10.12).

**Q 6.** Using Equation (10.14), explain how the conductivity of a semiconductor changes at high temperatures.

**Q 7.** What is the difference between Fermi-Dirac and Boltzmann distributions? Which distribution is being followed by the majority carriers in semiconductors?

**Q 8.** Given that the effective masses of electrons and holes in Si are approximately 1.08 \( m_e \) and 0.60 \( m_e \), respectively, and the electron and hole drift
mobilities at room temperature are 1350 and 450 cm²V⁻¹s⁻¹, respectively, and the energy band gap value is 1.10 eV, calculate the intrinsic concentration and intrinsic resistivity of Si [2].

Conduction in extrinsic semiconductors

In doped semiconductors, the dopant concentrations \( n_x \approx N_d \) for n-type and \( n_h \approx N_a \) for p-type doping at room temperatures are greater than the thermally generated intrinsic carrier concentrations \( n_i \). The conductivity depends on the carrier concentrations and the mobility. So to determine the temperature dependent conductivities, one has to consider, separately, how temperature affects both the carrier concentration and the mobility [2].

Temperature dependence of charge carrier concentration

Consider an n-type semiconductor with dopant carrier concentration \( N_d \) of arsenic atom (As). The As atoms introduce a donor energy level \( E_d \), that is located at a gap \( \Delta E \) below \( E_C \). The ionization of As atoms leads to electrons jumping across \( \Delta E \) into the conduction band. The scenario is depicted in Figure (10.5).

![Diagram](image)

**Figure 10.5:** Electron concentration of an n-type semiconductor in (a) low temperature regime, (b) medium temperature regime, (c) high temperature regime. \( E_f \) and \( E_d \) are the Fermi and donor atom energy levels, respectively.

1. **Low temperature regime.** At very low temperatures, conductivity is almost zero because donor atoms are not ionized due to the small thermal vibrational energy. As temperature slightly increases, the donor atoms get ionized and move to the conduction band as shown in Figure (10.5a). The
Figure 10.6: The temperature dependence of the electron concentration in an n-type semiconductor, showing the ionization, extrinsic and intrinsic regimes. Note that the horizontal axis is $1/T$ instead of $T$.

The electron concentration at such low temperature is given by,

$$n_e = \left( \frac{1}{2} N_c N_d \right)^{1/2} \exp \left( -\frac{\Delta E}{2k_B T} \right).$$

(10.15)

where, $\Delta E = E_c - E_d$ is the energy difference from donor energy level to bottom of conduction band. The low temperature regime is also called the ionization regime.

**Q 9.** What are the similarities and differences between Equations (10.12) and (10.15)?

**Q 10.** Explore the origin of the extra factor of one half in Equation (10.15).

2. Medium temperature regime In this temperature range, the process of ionization has continued to the extreme that all donor atoms have been ionized as shown in Figure (10.5b). This temperature range is often called the extrinsic range and is also indicated in Figure (10.6). Since the electrical conductivity depends on carrier concentration $n$ and mobility $\mu$,

$$\sigma = q n \mu.$$ 

(10.16)

and $n = N_d \propto$constant in the extrinsic region, the conductivity is solely determined by the temperature variation of the mobility. The mobility is proportional to some power $\alpha$ of the temperature.

$$\sigma = T^\alpha.$$ 

(10.17)

A plot of $\log(\frac{1}{\tau})$ versus $\log(\frac{1}{T})$ will give the value of temperature dependent coefficient $\alpha$. Extrinsic semiconductors are almost always operated in this region.
3. **High temperature regime** As temperature increases, the electron concentration \( n \) due to thermal agitations across the band gap is much larger than the dopant concentrations \( N_d \). In this regime, excitations from valence band to conduction band are also possible due to which hole concentration becomes equal to the electron concentration \( n_h = n_e \) depicted in Figure (10.5c). This range is referred as the intrinsic range and is shown in Figure (10.6). This is the regime where the purpose of doping is defeated and the material behaves as an intrinsic semiconductor. In this temperature range, the slope of \( \log(n) \) versus \( 1/T \) yields \( -E_g/2k_B \).

**Q 11.** A silicon ingot is doped with \( 10^{16} \) As atoms/cm\(^3\). Find the carrier concentrations and the Fermi level at room temperature (300 K). Also draw band diagram showing the Fermi level \( E_F \) and intrinsic Fermi level \( E_i \). [4].

**Q 12.** An n-type Si sample has been doped with \( 10^{15} \) phosphorus atoms cm\(^{-3}\). The donor level for P in Si is 0.045 eV below the conduction band edge energy [2]. (a) What would be the temperature above which the sample behaves as intrinsic? (b) What is the lowest temperature above which most of the donors are ionized?

**Temperature and impurity dependence of drift mobility**

Now that we have established how temperature affects carrier concentration, we turn attention to the mobility. Drift mobility \( \mu \) determines the average velocity \( v_d \) in the presence of an applied external field. The variation with temperature follows two distinct regions.

1. **High temperature region** Let suppose an electron in the conduction \( (C_B) \) or valence band \( (V_B) \) suffers collisions from a scattering ion \( (As^+) \). These scattering events depend on how strongly the ions vibrate, the amplitude depends on the temperature \( T \). The mean free time \( \tau \) between scattering events, is given by,

\[
\tau = \frac{1}{5v_t h N_a} \quad (10.18)
\]

According to the Drude model [2], the drift mobility is,

\[
\mu = \frac{e\tau}{m_e^*} \quad (10.19)
\]

In Equation (10.18), \( S \) is the cross-sectional area of the scatterer shown in Figure (10.7a), \( v_{th} \) is the mean speed of the electrons, called the thermal
velocity and \( N_s \) is the number of scatterers per unit volume. Now both the scatterer amplitude \( a \) and the thermal velocity of the electron \( v_{th} \) is temperature dependent. We unveil these dependences, one by one. The scatterer amplitude increases with temperature as \( a^2 \propto T \). Now an electron in the conduction band has only kinetic energy and the mean kinetic energy per electron in the conduction band is \( \frac{3}{2}k_B T \). Applying kinetic molecular theory to the gas of electrons in the conduction band, we obtain,

\[
\frac{1}{2}m^*v_{th}^2 = \frac{3}{2}k_B T.
\]

(10.20)

implying \( v_{th} \propto T^{1/2} \). Using the above derived temperature dependences of \( v_{th} \), the scatterer mean time \( \tau_L \) due to lattice vibrations will become,

\[
\tau_L = \frac{1}{SV_{th}} \propto \frac{1}{T^{3/2}} = T^{-3/2},
\]

resulting in lattice vibration scattering limited mobility, \( \mu_L \).

\[
\mu_L \propto T^{-3/2}.
\]

(10.22)

Clearly as the temperature goes up, \( \mu_L \) decreases.

2. **Low temperature region** At low temperatures, the scattering of electrons by thermal lattice vibrations is not strong enough. The electron scattering is performed by the electrostatic interaction with the ionized donor impurities. Let us consider a case in which an electron passes by an ionized donor \( \text{As}^+ \). The deflection from the rectilinear path depends on the following factors,
If the \( \text{K.E} \) of the electron is larger than the \( \text{P.E} \) of the ionized donor impurity (\( \text{As}^+ \)) at a distance \( r \) (\( \text{K.E} > |\text{P.E}| \)), then the electron will not feel the \( \text{P.E} \) and will continue its course unhindered, unswayed.

If the \( \text{K.E} < |\text{P.E}| \), then the coulombic interaction energy is strong enough to deflect the electron. The two cases are depicted in Figure (10.8).

\[ \text{K.E} \approx |\text{P.E}| \text{ at } r = r_c, \text{ the critical radius at which the electron is just scattered, leading to,} \]

\[
\frac{3}{2} k_B T = \frac{e^2}{4\pi \varepsilon_0 \varepsilon_r r_c^2}.
\]

From which one can deduce the critical radius,

\[
r_c = \frac{e^2}{6\pi \varepsilon_0 \varepsilon_r k T}.
\]

Thus, the critical scattering radius \( r_c \) also possesses the inverse temperature dependence and decreases as temperature increases. By adding the value of \( r_c \) into scattering cross section, \( S = \pi r_c^2 \), one may infer that \( S \propto T^{-2} \). Therefore, the ionized impurity scattering limited mobility, \( \mu \), comes out as replacing the same argument given before Equation (10.22)

\[
\mu \propto \frac{T^{3/2}}{N_I}.
\]

where \( N_I \) is the ionized impurity concentration. Thus in the low temperature regime, \( \mu \) decreases with increasing ionized impurity concentration.
Figure 10.9: Temperature dependence of mobility including effects of both lattice and impurity scattering in the two temperature regimes.

The lattice limited and impurity limited regimes of the mobility are shown in Figure (10.9). It is observed from Equations (10.22), (10.25) and Figure (10.9) that the mobility has a divergent behavior with respect to temperature. At low temperature, mobility increases with increasing temperature, and starts decreasing as temperature increases in the high temperature regime.

Q 13. Calculate the temperature dependence of the mean free time $\tau$ between impurities and derive Equation (10.25).

**Temperature dependence of conductivity**

We have determined the temperature dependence of the carrier concentration and mobility for a doped semiconductor. Hence, the electrical conductivity in extrinsic semiconductors can be determined by combining the results of Figures (10.6) and (10.9), as shown in Figure (10.10).

10.3 The experiment

**Overview of the experiment**

In this experiment we will investigate the conductivity temperature variation of a semiconductor sample (the base region in the unijunction transistor UJT(2N2646) and pure germanium Ge). The conductivity is derived from the measured resistance while temperature is controlled by a controller and measured using a
Figure 10.10: Combined effects of mobility and carrier concentration.

The semiconductor sample is placed inside a sample cell. The sample cell is placed inside a flow cryostat which can be filled by liquid nitrogen. The cell is cooled by the cool vapor of N₂. Alternatively, the cell can be heated to elevated temperatures by passing current through a heater wire wound around it. The flow cryostat is sketched in Figure (10.11a) highlighting the various components.

The resistance of the semiconductor sample is measured by the four-probe technique [5], [6]. A constant current is passed through the sample. The current is generated by a constant current source. The voltage drop across the sample is acquired by a four-probe circuit and facilitated by a Labview programme. To measure the temperature of the sample, a K-type thermocouple is placed inside the sample cell and the temperature control is achieved by a multi-zone controller.

The measured resistance at different temperatures yields the conductivity versus temperature behaviour of the semiconductor sample, from which the energy band gap \( E_g \) and the temperature dependent coefficient of mobility \( \alpha \) can be calculated.

**Apparatus**

The experiment involves the following major components.
1. cryostat
2. sample cell containing the sample and wound with heater wire
3. temperature controller
4. thermocouple
5. voltmeter with high input resistance
6. constant current source
7. power supply for the constant current source
8. solid state relay (SSR)
9. data acquisition system (DAQ)
10. supply of liquid nitrogen $N_2$.

Following some brief description of the above listed components. Our samples are a dice of pure Ge (donated by Dr. S. A. Siddiqi (Punjab University) and...
extremely expensive!) and the base of a unijunction transistor UJT, which is a lightly n-type material.

**Cryostat**

The cryostat, shown in Figure (10.12a) is a copper cylinder (4 inch in diameter, 12 inch long with one end closed), used to maintain cryogenic temperatures, and also able to withstand high temperatures. An insulating sheet is wrapped around the cylinder for reducing heat losses.

![Cryostat images](image)

Figure 10.12: (a) Cryostat. (b) Sample cell with heat sink.

**Sample cell**

For high temperature measurements, the semiconductor sample is slowly heated inside the sample cell wound with nichrome heater wire (Nichrom 37). It is essentially a copper pipe (20 mm diameter with one end closed). For low temperature measurement, a copper braid is attached to the bottom of the sample cell that serves as a heat sink, and is shown in Figure (10.12b). The heat sink ensures good thermal contact with liquid nitrogen. The sample cell is lowered into the cryostat for the measurements.

**Thermocouple**

A thermocouple is a temperature sensor used to measure the sample temperature. It is made up of two different metals with the metals at one end joined together.
Whenever the junction of two metals is heated or cooled, voltage is produced that can be correlated with temperature through a calibration procedure. In this experiment, we use a K-type thermocouple, for which some details are given in Table (10.1). The thermocouple reading is fed into a multi-zone controller.

<table>
<thead>
<tr>
<th>temperature range</th>
<th>-200 °C to 1250 °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>composition</td>
<td>Chromium-Aluminium</td>
</tr>
<tr>
<td>sensitivity</td>
<td>~ 41 μV/°C</td>
</tr>
</tbody>
</table>

Table 10.1: Some characteristics of the K-type thermocouple.

**Temperature controller**

Our aim is to attain a stable temperature for the sample, for which a multi-zone controller (CN1504-TC from Omega Engineering) is used. It is a compact unit comprising four PID controllers, and is shown in Figure (10.13). Additionally, it is also equipped with an ON/OFF control [8]. A temperature profile is programmed into the controller to achieve a particular temperature for a certain duration of time. To obtain an understanding of the programming modes and a guide to constructing the profiles, refer to Appendix A or the datasheet of the controller [8].

![Temperature controller](image)

Figure 10.13: Temperature controller, (a) front view, (b) rear view.

The advantage of the temperature controller is that it possesses a “Proportional, Integrator, Derivative” (PID) controller. The PID controller calculates an error value as the difference between a measured process variable and a desired set point. For more details see Appendix B.

Q 14. Read Appendix A. Programme a profile in the temperature controller.
Your starting point will be the room temperature, achieve 40°C in 1 min and keep the temperature fixed at 40°C for 10 minutes. Subsequently, migrate to 50°C in 1 min and keep the temperature fixed at 50°C for 10 minutes.

The purpose of the previous step is to familiarize you with the programming of the temperature controller. **This step must be completed and practiced before the start of the experiment.**

**Four-probe versus two-probe method**

Resistance measurements are often taken by a two-probe method, that is depicted in the self-explanatory Figure (10.14a). The disadvantage of this method is that some voltage drop occurs across the lead resistance, hence the voltage measured by the meter ($V_m$) will be different from the actual voltage ($V_z$) across the test resistance. So when applied to low test resistances ($R_z < 100$ Ω), the voltage drop due to current ($I_{lead}$) flowing through the lead resistance can no longer be neglected, leading to significant error in the results.

In order to remove the error of contact resistances, a four-probe method is preferred, specially when $R_z$ is small. In this method, current is passed through the test resistance using one set of leads and voltage drop is measured through another set, as depicted in Figure (10.14b). Although small current may flow through the voltage measuring leads but that would be much smaller, owing to the extremely large input resistance of the voltmeter, and can generally be neglected for all practical purposes [6].

**Constant current source and voltmeter**

The resistance of the semiconductor sample is measured by passing constant current through the sample and measuring the voltage drop generated across it. In order to provide stable current to the sample, a constant current generator is required and is shown in Figure (10.14c). **You will be required to construct this constant current source.** The sample resistance ($R_x$) is calculated from the output voltage as,

$$R_x = \frac{V_o}{I} \quad (10.26)$$

$$= R(\frac{V_o}{V_z}), \quad (10.27)$$

where $V_z$ is the Zener voltage and $V_o$ is the output voltage that will ultimately
Figure 10.14: The Electronic circuitry. (a) 2-probe method, (b) 4-probe method, (c) current source, (d) overall circuit showing the current source and differential amplifier, (e) DC power supply.

provide the sample resistance given in Table (10.2). Figure (10.14c) shows a circuit that generates 1 mA current. The operational parameters for the various samples that will be used in this experiment, given in Table (10.2).
### Table 10.2: Current (I) required for semiconductor samples.

<table>
<thead>
<tr>
<th>Semiconductor sample</th>
<th>$R_x$</th>
<th>$I$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unijunction transistor (UJT)</td>
<td>up to 10 kΩ</td>
<td>100 μA</td>
</tr>
<tr>
<td>Germanium (Ge)</td>
<td>&lt;1 kΩ</td>
<td>10 mA</td>
</tr>
</tbody>
</table>

The overall four-probe circuit including a current source and a voltmeter, based on a differential amplifier [10], is drawn in Figure (10.14d). Two separate contact pairs are made on both ends, (AB) for passing current and (CD) for voltage detection, as indicated in Figure (10.14d). The output from the voltmeter is fed into the data acquisition system (DAQ).

**Q 15.** Make circuits capable of providing stable currents of 100 μA and 10 mA.

**Q 16.** What is the gain of the differential amplifier circuit given in Figure (10.14d)?

**Q 17.** What is the advantage of a differential amplifier as compared to an ordinary amplifier.

### Power supply

In order to supply DC voltage to the constant current source and voltmeter, a bipolar power supply is used in which AC voltage is converted to a selectable DC voltage. The home built power supply is already provided to you. For interested students, the circuit description of the supply is presented in Figure (10.14e). The assembled circuit is photographed in Figure (10.15a) and enclosed in an aluminium box depicted in Figure (10.15b). The BNC connectors provide +12 V, −12 V and ground points.
Experimental procedure

1. Make circuits for constant current sources of 100 μA, 10 mA and the voltmeter on bread board according to the circuit diagrams in Figures (10.14c), (10.14d).

2. You are provided with one unijunction transistor (UJT) and a dice of pure germanium (Ge). Current and voltage wires have already been connected using silver paint with these samples, shown in Figure (10.16).

![Current leads](image1)
![Ceramic rod](image2)

**Figure 10.16:** (a) Pure germanium Ge. (b) Unijunction transistor UJT.

3. Pass the insulated wires, enshrouded in a ceramic rod through the steel pipe. Connect current and voltage leads with constant current source and voltmeter. You will acquire voltage through the data acquisition system (DAQ).

4. Before making any electrical connection, set the output voltage of the variac to 100 V. Measure it with a voltmeter.

5. Layout the experiment as suggested in Figure (10.11). Do not switch on power at this stage.

6. Connect the solid state relay (SSR) with the sample cell as well as with the temperature controller, in a way that its positive terminal is connected to the positive output channel of the temperature controller, and negative terminal is connected with the negative output channel of the controller. The remaining terminals are connected to the sample cell and variac, as sketched in Figure (10.11b).

7. Open the Labview file titled **energyband.vi**. Make a folder, and enter your file name (e.g D:\amrozi\test.lv). Your data will automatically be saved
in that file. You will use the DAQ system together with the Labview code to measure the output voltage of the four-probe circuit.

8. First set the variac at 100 V. Switch on the variac, the temperature controller will turn on. Leave the setup for 10 minutes to warm up the temperature controller.

9. Turn on the DC power supply that is connected to the current source and voltmeter. Check the output of the voltmeter.

10. **UJT sample, low temperature measurement:**

    - It will not be required to programme the temperature controller in this phase of the experiment as the temperature is naturally increasing.
    - Add liquid nitrogen to the cryostat through the funnel.
    - The temperature controller will start showing the corresponding temperature. Pour liquid $N_2$ until you reach the temperature $\sim (-150^\circ C)$.
    - Leave the setup to warm up to room temperature. As it warms, do the following.
    - Take voltage measurement after every $10^\circ C$ rise in temperature using the Labview file `energyband.vi`.
    - Suppose, when the temperature controller shows $-150^\circ C$. In the front panel, enter the value $-150^\circ C$ in the temperature box. Press the run button to start acquiring data and the stop button to stop the acquisition, while the temperature is still $-150^\circ C$.
    - Open Matlab, and add the appropriate folder to the Matlab path.
    - Take the mean of the voltage value at the different temperatures.

11. **UJT sample, high temperature measurement:**

    - When you have reached room temperature, you will start taking your readings for higher temperatures. For this purpose, you will require the use of the heater and the PID controller.
    - Suppose you want to take readings in steps of $10^\circ C$, ranging from $40^\circ C$, $50^\circ C$, up to $250^\circ C$.
    - Suppose the room temperature is $32^\circ C$.
    - Now we need to enter a profile so that the temperature negotiates $40^\circ C$, $50^\circ C$ and $60^\circ C$, as it is only possible to programme for these three points at one time. Now enter this temperature profile according to the instructions in Appendix A.
Acquire the voltage for 40°C, 50°C and 60°C using Labview as discusses earlier.

For the next set of points (e.g. 70°C, 80°C, 90°C), you will have to repeat the above steps.

Complete your measurements up to ≈ 250°C.

Find resistance from the acquired voltage.

Tabulate temperature versus resistance, as indicated in Table (10.3).

<table>
<thead>
<tr>
<th>Temperature (°C)</th>
<th>Resistance (Ω)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-150</td>
<td></td>
</tr>
<tr>
<td>-140</td>
<td></td>
</tr>
<tr>
<td>.</td>
<td></td>
</tr>
<tr>
<td>.</td>
<td></td>
</tr>
<tr>
<td>250</td>
<td></td>
</tr>
</tbody>
</table>

Table 10.3: Relationship between temperature and sample resistance.

12. Ge sample, high temperature measurement:

- You are provided with a pure Ge sample, an intrinsic semiconductor. Set up the complete experiment being guided by your experience with the UJT sample. The constant current for the Ge sample is \( I = 10 \text{mA} \).

- Again enter the temperature profile in the temperature controller. You will require measurements up to 150°C.

- Complete acquiring the data.

- Tabulate temperature versus resistance.

**Q 18.** Plot a graph of \( \log(\frac{1}{n}) \) versus \( T \) in Kelvins, and distinguish the intrinsic and extrinsic regions for the UJT.

**Q 19.** Calculate the energy band gap from the intrinsic region data for UJT and Ge. As conductivity depends on both the charge carrier concentration and mobility, so in calculating band gap take both of these factors into account. The published energy band gap values for silicon (Si) and germanium (Ge) are provided in Table (10.4).

**Q 20.** Calculate the temperature coefficient \( \alpha \) of the carriers mobility from the extrinsic region data of the UJT. The published value of \( \alpha \) is 2.3, given in
<table>
<thead>
<tr>
<th>Semiconductor</th>
<th>Band gap</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silicon (Si)</td>
<td>1.12 eV</td>
</tr>
<tr>
<td>germanium (Ge)</td>
<td>0.67 eV</td>
</tr>
</tbody>
</table>

Table 10.4: Energy band gap ($E_g$) for Si and Ge.

[10].

**Q 21.** What is your uncertainty in the energy band gap measurement?

**Q 22.** In the low temperature measurements, the sample is continuously thawing (warming) as data is being acquired. What kind of uncertainty does this cause in your measurements?

## 10.4 How to construct a temperature profile in temperature controller

Suppose we want to make the temperature profile, sketched in Figure (10.17).

For this you have to do the following steps.

![Temperature profile diagram](image)

Figure 10.17: An example of temperature profile. Setpt 1, 2 is 40°C, setpt 3, 4 is 50°C, and setpt 5, 6 is 60°C.

1. **The very first and the most important step is to power up the temperature controller.** Power connections should be made to L1, L2. L1 is connected to the positive terminal of the AC power supply and L2 with the negative one. Our AC power supply is the variac, and it should be set at 100 V. Be careful in applying voltage to the temperature controller as
it can be damaged, if you operate it over 120 V AC.

2. To enter into the programming mode, hold the **CTR.SEL** key until the unit displays **EntEr PASSCoDe**. At this point, enter the pass code which is 3254 for our device.

3. Select programme system **PrG SYS mode**, the controller will enter into the system configuring mode. Push **PROG** key, the display will show **dSP OPT**, select **Process-Setpoint**. You have to push **PROG** key to go to the next mode. The display will show **dSPLy t**, enter time in seconds using **▲▼** and < DIG > keys. The **▲▼** and < DIG > keys are used to assign values. Select **Stt SP** mode. Assign a value of 10 to **rAtE tb** mode. Enter **COLD JN** value that would be the room temperature. Enter a value of 10 to the **V rAnGE** mode and next to it, assign value to the **C RAnGE** that would be 20.

4. Move to the programme control **PrG Ctr** mode. Push **PROG** key, the display shows **CntrL x** (where x=controller no.), select controller 1. Then go to **Ct tYPE** mode and select **PID** option. Select **Cr.Al** (for the K-type thermocouple) for **SIgnAL** mode. Then select **dEGrE C** and finally, select engineering unit mode **Cx UNIt** that would be **C** if input signal is a thermocouple.

5. Next move to the tuning mode. Push **PROG** key until the unit displays **tUning**. Select the **PID** mode. Enter the following values as,

- proportional band = 0.5,
- reset = 1,
- rate = 1.

6. Push **PROG** key to enter into profile mode. When the unit displays the **PROFILE**, select the required controller (controller 1) using **PROG** key. When the display will briefly read **Stt SP**, enter the value of starting set point that would be room temperature, for example 32°C as indicated in Figure (profile). Use **▲▼** and < DIG > keys to enter the desired value. Next press **SETUP** to go to the next function. At this point the display will briefly read **SetPt 1** enter the value of setpoint 1, for example 40°C. Push **PROG** key to go on to the next step. The display will show **EntEr t**, enter the time value like 1 min. Move to setpoints 2 by pressing **PROG** key. Enter setpoint 2 value which is 40°C and time=10 min. Repeat the above steps to program up to setpoint 6 as shown in Figure (10.17).

\[\text{BIBLIOGRAPHY}\]
7. Push EXT key. Again push EXT key for saving programme. Select the desired controller using PROG key and run it by pressing RUN/STOP key.

10.5 PID controller

The PID controller manipulates the process inputs based on the history and rate of change the signal [9]. The basic idea is that the controller reads the system by a sensor as shown in Figure (10.18a). Then it subtracts the measurement from the desired reference to generate the error value. The error will be managed in three ways: to handle the present through the proportional term; recover from the past using the integral term; and to anticipate the future through the derivative term.

- **P** controller: $K_p e(t)
- **I** controller: $K_i \int_0^t e(t) dt
- **D** controller: $K_d \frac{de(t)}{dt}

![PID controller block diagram](image)

(a)

![PID controller response](image)

(b)

Figure 10.18: (a) Block diagram of a PID controller. (b) Response of a PID closed loop system.
Proportional term

The proportional term (sometimes called gain) changes the output in proportion to the present error value. The proportional response can be adjusted by multiplying the error by a constant \( K_P \) called the proportional gain. A high proportional gain results in a large change in the output. If the proportional gain is too high, the system can become unstable. In contrast, a small gain results in a small output response, and a less responsive controller.

Integral term

The integral term (sometimes called reset) is proportional to both the magnitude and duration of the error. This term sum up the previous errors to the system control input. The summing of the error will continue until the system process value equals the desired value, and this results in no stationary error. The magnitude of the contribution of the integral term correction is determined by the integral gain \( K_I \).

Derivative term

The derivative term (also called the rate) calculates the rate of change of the process error by determining the slope of the error over time. The magnitude of the contribution of the derivative term to the overall system is determined by the term \( K_D \). The derivative term slows the rate of change of the controller output. However, derivative control is used to reduce the magnitude of the overshoot produced by the integral term and improve the controller-process stability. If the derivative term is too large, it can cause the process to become unstable as this term is highly sensitive to noise in the error.
Chapter 11

Tracking Brownian Motion through Video Microscopy

Asma Khalid and Muhammad Sabieh Anwar

Brownian motion is the random motion of colloidal particles suspended in water, air or any other solvent. In 1905, Einstein argued that this motion is a direct evidence for the atomic nature of matter. Einstein’s and Perrin’s efforts helped raise the status of atoms from useful hypothetical objects to objects whose existence could no longer be denied.

KEYWORDS: Brownian motion · Colloidal particles · Viscosity · Boltzmann constant · Image processing · Kinetic theory

APPROXIMATE PERFORMANCE TIME: 1 week

11.1 Objectives

In this experiment, we will,

1. observe Brownian motion of microparticles,
2. calibrate a compound microscope,
3. use the microscope’s built-in camera to record Brownian motion,
4. learn how to extract images and frames from a movie using MATLAB,
5. use some basic and simple commands for image processing,
6. locate and track microparticulate motion.

7. plot Brownian motion in 2-D.

8. observe how the mean square displacement of particles helps calculate Boltzmann’s and Avogadro’s constants.
Bibliography


[10] Motic Live Imaging Module, Quick start Guide


11.2 Theoretical Introduction

In 1827, Robert Brown observed the random motion of micro-particles suspended in gases and liquids. He called this random or jiggling motion 'Brownian motion'. However, it was only in 1905 that Einstein first explained this phenomenon on the basis of kinetic theory of molecules. In a quantitative manner, Einstein connected quantities of kinetic theory such as viscosity and mobility with the Brownian motion.

Einstein performed a statistical analysis of molecular motion and its effect on particles suspended in a liquid. As a result, he calculated the mean square displacement of these particles. He argued that an observation of this displacement would allow an exact determination of atomic dimensions and prove the existence of atoms and verify the molecular kinetic theory of heat.

Perrin, a brilliant experimentalist, performed a series of experiments in the first decade of the twentieth century, one of which depended on Einstein's calculation of the mean square displacement of suspended particles. His results confirmed Einstein's relation and thus the molecular-kinetic theory. Eventually a physical explanation of the phenomenon of Brownian motion led to the acceptance of the atomic or molecular-kinetic theory [1].

Brownian motion and kinetic theory

Brownian motion can be explained using the kinetic theory of matter and the kinetic molecular theory of heat. The kinetic theory of matter posits the existence of atoms and molecules, and their constant motion due to which they elastically collide with one another. The kinetic molecular theory describes temperature as the constant motion of atoms and molecules in matter.

Brownian motion which is characterized by the constant and erratic movement of minute particles in a liquid or a gas is thus due to the inherently random motions of the atoms or molecules that make up the fluid in which the particles are suspended. The fluidic atoms or molecules collide with the larger suspended particles at random, making them move randomly.

Einstein described that Brownian motion actually arises from the agitation of individual molecules due to the thermal energy $k_B T$ they possess at a specific temperature. The collective impact of these molecules against the suspended
particle yields enough momentum to create movement of the particles.

Mathematical picture

The origin of Brownian motion can be understood on the basis of the theorem of equipartition of energy [2]. Each colloidal micro particle, possessing a mass \( m \) is free to exhibit translational motion. The mean kinetic energy of the particle in three dimension is,

\[
\frac{1}{2}mv^2 = \frac{3}{2}k_B T.
\]

This energy, though small in value, leads to a measurable amplitude of vibration for a small micro particle. It is worth noticing that in addition to the random fluctuating force, the particles also experience a drag force (frictional force) as they are pulled through the solvent.

To find a solution to the motion of the particles, we will use the Langevin equation for a particle of mass \( m \) and velocity \( \mathbf{v} \)

\[
m\frac{d\mathbf{v}}{dt} = -\alpha\mathbf{v} + \mathbf{F}(t).
\] (11.1)

From Equation (11.1), we can see that each colloidal particle is subject to two forces:

1. the random molecular bombardment \( \mathbf{F}(t) \) that causes the Brownian motion, and the

2. resistive force \( \alpha\mathbf{v} \), where \( \alpha \) is the damping coefficient related to viscosity of the fluid or solvent.

In one dimension, the scalar form of Equation (11.1) is written as,

\[
m\frac{d^2x}{dt^2} + \alpha \frac{dx}{dt} - F(t) = 0.
\]

Multiplying both sides of the above equation by \( x \), yields,

\[
xm\frac{d^2x}{dt^2} + \alpha x \frac{dx}{dt} - x^2F(t) = 0. \quad (11.2)
\]

To simplify, we use the expansion of the expression \( \frac{d^2}{dt^2}(x^2) \). The modified Equation (11.2) becomes,

\[
\frac{m}{2} \frac{d^2x^2}{dt^2} - \frac{m}{2} \left( \frac{dx}{dt} \right)^2 + \frac{\alpha}{2} \frac{dx^2}{dt} - xF(t) = 0. \quad (11.3)
\]
Q 1. Using chain rule, expand the derivative $\frac{d^2}{dt^2}(x^2)$ to obtain Equation (11.3).

Now we use the theorem of equipartition of energy to find the average energy of a single particle for one degree of freedom, which is given by

$$\frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} k_B T.$$  

$$\frac{m}{2} \langle \frac{d^2}{dt^2} \rangle = \frac{1}{2} k_B T. \quad (11.4)$$

We average Equation (11.4) over time and recognizing that since $F$ is a random force, hence $\langle xF \rangle = \langle x \rangle \langle F \rangle = 0$. Defining $\beta = \langle \frac{d^2}{dt^2} \rangle$ and substituting Equation (11.4) into (11.4), we obtain,

$$\frac{m}{2} \frac{d\beta}{dt} + \beta = 0. \quad (11.5)$$

Q 2. Derive Equation (11.5) (refer to [4] and [5] for help) with respect to time and show that the solution is,

$$\beta = \frac{2k_B T}{\alpha} + \exp(-t\alpha/m), \quad (11.6)$$

where $A$ is the integration constant. For a reasonably long observation time ($t = \tau$), the factor $-t\alpha/m$ will be very small and hence we can ignore the second term on the right hand side. Finally integrating the modified Equation (11.6) over the observation time $\tau$, we get

$$\int_0^\tau \langle \frac{d^2}{dt^2} \rangle dt = \int_0^\tau \frac{2k_B T}{\alpha} dt$$

$$\langle x^2 \rangle = \frac{2k_B T \tau}{\alpha}. \quad (11.7)$$

For spherical particles, each of radius $a$, Stokes law can be used to write $\alpha$,

$$\alpha = 6\pi \eta a,$$

where $\eta$ is the viscosity of the fluid. Equation (11.7) hence takes the form,

$$\langle x^2 \rangle = \frac{2k_B T \tau}{6\pi \eta a}.$$

Using above Equation, we can write the mean squared displacement in two dimensions,

$$\langle r^2 \rangle = \frac{4k_B T \tau}{6\pi \eta a}. \quad (11.8)$$

Hence, by plotting $\langle r^2 \rangle$ as a function of time, we expect a straight line through the origin whose slope can be used to obtain Boltzmann’s constant $k_B$. Equation
(11.8) is traditionally written as,

\[ \langle r^2 \rangle = 4DT. \]  \hspace{1cm} (11.9)

where \( D = k_B T/(6\pi \eta a) = k_B T/\alpha \) is the self-diffusion constant [1], [2].

Using the relation between Boltzmann’s constant and molar gas constant \( R \), we can also use Equation (11.8) to find Avogadro’s number \( N_A \).

\[ N_A = \frac{1}{\langle r^2 \rangle} \frac{2RT}{3\pi \eta a^2}. \]  \hspace{1cm} (11.10)

Q 3. Explain the dependence of the diffusion constant \( D \) on the damping factor \( \alpha \).

### Significance of Brownian motion

The theory of Brownian motion has come a long way since its humble beginnings in the nineteenth century. There are now a large number of applications that have evolved. Some of the applications are listed below.

- In electronic devices, the discussion of Brownian motion is specifically important in understanding the effects of thermal motion of electrons contributing to Johnson noise [6].

- Researches in the field of biomedicine have shown that Brownian motion plays a critical role in the transport of enzymes and chemicals both into and out of cells in the human body. Scientists have subsequently discovered that many fundamental processes in living cells are driven by Brownian motion which also shows potential for use as probes at the nanoscale. For example it is possible to obtain detailed information about a particle’s environment by analyzing its Brownian trajectory [7].

- In environmental sciences, the application of Brownian motion enables the prediction of extreme floods and droughts. By modeling volumetric flow of rivers mathematically, the relation can define a Brownian walk [8].

### 11.3 Apparatus

1. Motic microscope with built-in 3.0 megapixel camera and data transfer cable USB 2.0
2. Motic calibration slide
3. 76 mm x 25 mm glass slides
4. 18 mm x 18 mm cover slips
5. polystyrene microspheres (by Polysciences) diluted in water
6. 20μL micro pipette with tips

Figure 11.1: A diagram of the microscope.

**Apparatus handling**

The two main components of the experiment are the microscope and the polystyrene microspheres solution. The Motic microscope is a biological compound microscope, model BA210 with a 3.0 mega pixel built in camera. Live image resolution of the camera is 2048X1536 and data transfer rate is 480 MB/sec. The scope has four different objectives i.e., 4X, 10X, 40X and an oil immersion 100X objective. This range of different objective lenses allows us to select a suitable magnification to view the sample under observation. Microscope’s objectives are parfocal, which means that when the objective lenses are changes the sample stays in focus. The light source in the scope is a 30 Watt halogen bulb whose intensity can be varied with the intensity control knob. Different parts of the microscope are labeled in Figure 11.1. The microspheres which will be used in the experiment are made of polystyrene and are available in five different sizes (diameters), i.e., 0.75 μm, 1 μm, 1.5 μm, 2.0 μm, 4.0 μm. These microparticles are harmless and should be used with normal precautions and care. However,
special care should be taken for their storage. They should be stored at 4°C and in no case be permitted to freeze.

![Microscope with slide preparation components.](image)

**Figure 11.2:** Microscope with slide preparation components.

### 11.4 Experimental method

**Microscope adjustment and Image viewing software**

To start the experiment, the **Motic Image Plus 2.0** application is run, and the **Capture** window is opened. The built-in camera will execute a live image in the window. One of the three resolutions is selected, the white balance is applied, and the image is fit to the window.

**Obtaining and Focusing the Image**

1. Foremost, the calibration slide is fixed in the **specimen holder**.

2. An objective lens is selected with which calibration and all other measurements and recordings are to be taken.
3. Focusing is carried out with the **Coarse** and **Fine** focusing knobs. These knobs are present at the bottom left and right of the microscope. The direction of vertical movement of the stage corresponds to the turning direction of the focus knobs and one rotation of this knob moves the stage by 0.2 mm. For more details on setting up the microscope and obtaining a sharp focused image, refer to [9].

**Image calibration**

Calibration is a very important step to be performed prior to the recording of any data, since the image processing and particle tracking algorithm is strongly dependent on the particle size, hence any carelessness in this step can lead to erroneous data. Better the calibration, easier would it be to record the particles' trajectories. Another important thing is to record the movie using the same objective and image resolution with which the calibration was done since each combination of objective and image resolution give different calibration values. The 10X and the 40X objectives are most suitable for our microparticles. Refer to [10] and [11] for detailed calibration methods using the **Motic** calibration...
slide. The calibrated values are in terms of μm per pixel and using this the we can acquire the microspheres' diameter in microns.

**Microsphere solution**

To observe Brownian motion we use polystyrene microbeads. These microbeads (or microspheres) are available in the lab in a range of different diameters (0.75 to 4.00 μm). Refer to [13] for the microspheres’ safety data sheets and physical properties. When originally packaged, the beads are suspended in 5 or 10 mL of water. Their concentration is very high. The estimated mass of microbeads in 1 cm³ is 1.05 grams. Due to the high sphere density the solution is diluted 10⁶ times. This increases the average distance between the beads so that the spheres do not frequently collide with each other and can be independently visualized and tracked.

A diluted solution has already been prepared for you and provided in the squeezed bottle. This solution should be shaken well before use, since the spheres start to settle down at the bottom with the passage of time.

**Observation slide preparation**

1. A custom observation slide of dimensions 76 mm x 25 mm and coverslip 18 mm x 18 mm is used for sample preparation. Before use both the slide and the coverslip are washed with acetone to remove any grease or dust on the surface.

2. The rotating knob of micro pipette is set to obtain a volume between 2 – 5 μL. A tip is fixed to the pipette's narrower end. A drop of the diluted microsphere solution is sucked by pressing the head of the pipette gently and released by pressing the head with a little force of thumb. The drop is then placed on the slide.

3. Now the cover slip is placed on the slide and gently pressed on to it to form a seal between the slide and the coverslip.

4. Two thin slices of plastic tape or paper are stuck to the slide in order to prevent the adhesion of liquid to the glass surface as shown in Figure 11.4. This unnecessary adhesion can stick the microspheres to the surface of glass, hindering their free movement and leaving them immobile.
5. The glass slide is placed on the microscope's specimen stage and fixed using the specimen holder. The image is viewed on the computer screen. If there is a coherent macroscopic flow in the liquid, the slide should be discarded and a new one should be prepared.

![Diagram of slide mounting system](image)

Figure 11.4: (a) Illustration of slide mounting system, (b) An observation glass slide with microsphere solution. A cover slip is used to prevent the liquid flow due to air currents.

Figure 11.5 shows different steps involved in preparation and mounting of the sample slide. Figure 11.5 (a) shows pouring a 5 μL diluted solution drop on the slide, (b) illustrates placing the glass cover slip on the slide whereas (c) shows the slide being mounted on the microscope's stage.

![Photographs of slide preparation](image)

Figure 11.5: (a) Using micro pipette to place the solution of a known volume at the center of the slide, (b) covering the slide with cover slip, (c) mounting the slide onto the specimen stage.

**Recording Brownian motion**

Once an acceptable sample slide is obtained, the **Record** button in the *Capture* window is pressed, and the destination for saving the file is selected (preferably MATLAB's work folder). The software will start recording a movie of the Brownian motion. While recording data, the following precautions must be followed,
1. The microscope or slide should not be subjected to any movement.

2. The stage and the slide shouldn’t be exposed to any air currents.

3. The intensity of surrounding light and light coming from the lamp of the microscope must not change.

4. Avoid storing movies longer than 15-20 seconds interval. Adjust the time length according to the resolution of image and frames per second. Number of frames per second is large for smaller resolution and is lower for higher resolutions.

5. Take at least five movies from a single slide. These movies are of course saved with different names.

**Particle tracking through MATLAB**

For data extraction from the movie, image processing and particle locating, the MATLAB file `tracker.m` is run. The following sections give a description of this program file. You are encouraged to study the source code and convince yourself of the various data extraction, image processing and particle tracking tasks described here.

**Extracting movie frames**

1. The recorded Brownian motion movie saved in the MATLAB’s work folder is loaded into MATLAB file using the command

   \[
   \text{obj} = \text{mmreader('filename.avi')}. \]

   This command can read in video data from a multimedia file and save it in an object `obj`. This object will be specified by the duration of the movie, frame rate, total number of frames, height, width and video format of the movie.

2. The number of frames per second (fps) is usually large, normally greater than 20 fps. In this case the built-in camera’s resolution is not fast enough to detect the particle’s movement every \((1/20)\)th of a second, hence, a suitable step size needs to be input. The step size can be any integer lying in the range 5-20, depending on the number of frames available. If the total number frames are 100, the frame rate is 10 fps and we select the step size
to be 5, for example, then our program will select every fifth frame, thereby providing only 2 frames per second and a total of 20 frames. The step size helps discard similar frames and ensures that MATLAB does not run **OUT OF MEMORY**.

3. **obj** is an \( H \times W \times B \times F \) matrix where \( H \) is the image frame height, \( W \) is the image frame width, \( B \) is the number of bands in the image (e.g. 3 for RGB), and \( F \) is the number of frames read in. The following command

\[
\text{images}(:,:,\text{count}) = \text{read}(\text{obj}(:,:,n))
\]

is used to read in video frames from **obj** one by one and save them as separate images. Here **count** specifies the serial number of the frames to be saved and **n** is the index running from first to last frame with the specified step size. For example, if \( n = 1 : 5 : 200 \), then the first frame will be saved as \( \text{images}(:,:,1) \), sixth frame as \( \text{images}(:,:,2) \), eleventh frame as \( \text{images}(:,:,3) \) and so on. The type of data returned is always **UINT8** data representing RGB24 video frames.

4. The images are cropped if required.

**Image processing**

To differentiate the background, dust particles and water droplets from the microspheres, the following steps are performed on images to make the microsphere tracking more accurate and reliable.

1. The **RGB2GRAY** command is used to converts RGB images to grayscale by eliminating the hue and saturation information while retaining the luminance.

2. **IMCOMPLEMENT** computes the complement of the images. This converts the dark particles to bright white particles with maximum brightness.

3. **IM2BW(image, LEVEL)** produces binary images by converting the gray images to binary by thresholding. The output binary image has values of 1 (white) for all pixels in the input image with luminance greater than **LEVEL** and 0 (black) for all other pixels.

4. Selecting **LEVEL**: A threshold level is necessary to choose for converting gray images to binary. The value of this variable lies preferably in the range 0.4 to 0.7. To select a threshold level, following steps are performed
• A value of **LEVEL** (lying in the range 0.4 – 0.7) is entered when prompted by the program.

• The result of applying the **LEVEL** to a typical frame will be displayed on the screen.

• The process is repeated until a suitable threshold level has been applied.

• To exit from the process, a random value lying outside the specified range is selected and the most suitable last value of **LEVEL** will be saved and will be used for all subsequent images.

**Note:** The value of **LEVEL** should be very carefully selected in a manner that the binary image so obtained shows maximum number of microspheres appearing as bright and well defined particles. Values with a difference of 0.01 can greatly improve the quality of the data. The original **RGB** image and its modification to binary images using three different **LEVEL** values are shown in Figure 11.6.

![Figure 11.6](image_url)

Figure 11.6: (a) Original image, in contrast to its binary forms with levels of threshold at (b) 0.4, (c) 0.5 and (d) 0.56.

**Q 4.** Why is Figure 11.6 (b) the most undesired binary counterpart of the original image?
5. A real-space bandpass filter \texttt{bpass.m} [14] is implemented on the binary images that suppresses pixel level noise and long-wavelength image variations while retaining information of a characteristic size. The function is called by the following command

\[
\texttt{filtered\_images} = \texttt{bpass(binary\_images, lnoise, lobject)}
\]

- \texttt{lnoise} is the characteristic length scale of noise in pixels and all the noise of this length would vanish from the image. One may assume any positive floating value, preferably in the range 0-1. This parameter is carefully selected in a way that all the noise lying in the neighborhood of a sphere vanishes and microspheres of a certain diameter are distinguished as spherical objects with sharp boundaries.

- \texttt{lobject} is the microspheres’ length in pixels and should be somewhat larger than the actual sphere’s diameter so that all particles with diameters in that range are counted. This value is taken from the image calibration and sphere size calculation performed in Section 11.4. All the particles that lie out of this range will be considered as background noise and will be subtracted from the final filtered image [14].

Figure 11.7 shows different bandpass filtered images with different values of noise parameter.

\textbf{Particle tracking and data extraction}

Once we have obtained filtered images in which microspheres appear as smooth, spherical and bright particles, the next step is to locate these bright intensity particles in every frame and then link the trajectory of each microsphere.

\textbf{Particle Finder} The program used for locating particles in a frame is named \texttt{pkfnd.m} [14] which finds local maxima in an image to pixel level accuracy and provides a rough guess of the particle’s center. The command to call this function is as follows

\[
\texttt{position} = \texttt{pkfnd(filtered\_images, int, d)}
\]

- \texttt{filtered\_images} can be any of the frames processed in Section 11.4.

- \texttt{int} is the minimum brightness of a pixel that might be a local maxima. Approximate this parameter by setting a lower limit for the maximum intensity
Figure 11.7: (a) Binary image, in contrast to its filtered forms with lengths of noise, Inoise at (b) 0, (c) 0.5 and (d) 0.7 and (e) 1. The value of Iobject for all of these images is 3. The boundaries of particles become more smooth and circular as the noise length is increased.

value, which is provided by the command $\max(\max(\text{filtered_images}))$ for any one of the frame.

- $d$ is the diameter of the particles in pixels, this input helps get rid of any particle found within a radius of $d/2$, or agglomerated particles, and the code will keep only the brightest.
Figure 11.8: A flow chart outlining the various operations in the particle tracking software.

The output _position_ is an $N \times 2$ matrix containing as many rows $N$ as there are particles in the image. The first column gives the x-coordinates and the second gives the y-coordinates of the positions of particles.

**Centroid Finder** After finding the positions of particles in a specific frame using the particle finder program, we then have to find the exact location of the centroid of each particle in all the frames. The program _cntrd.m_ [14] is called within a loop that runs from the first to the last frame. The following command is used

\[
\text{centroid \_location} = \text{cntrd(\textit{filtered \_images}, \textit{position}, \textit{dim})}
\]

which runs for all the frames and calculates the centroid of all bright spots in the frame to sub-pixel accuracy.
11.4. EXPERIMENTAL METHOD

- **position** is the locations of local maxima obtained from **pkfind.m**.

- **dim** is the diameter of the window over which to average to calculate the centroid. This value should be big enough to capture the whole particle but not so big that it captures other particles too.

Output of this function is an $N$x$4$ array containing x-coordinates, y-coordinates and brightness for each feature.

- **centroidLocation(:,1)** are the x-coordinates.
- **centroidLocation(:,2)** are the y-coordinates.
- **centroidLocation(:,3)** are the brightness.
- **centroidLocation(:,4)** are the square of the radius of gyration (radius of gyration is the root mean square distance of the microsphere’s parts from either its center of gravity or from an axis).

Now after having the coordinates of all the particles in all the frames, the only task left for you, for which you have to write your own program is to subtract the initial position of each particle in the first frame from the positions in the successive frames, to find the displacement. The displacement is then squared and averaged over the number of particles $N$ in each frame and then plotted against time.

The initial positions of all particles in first frame are taken as the origin $O(x_0,p,y_0,p)$, where $p$ runs from 1 to $N$.

The square of the distance of the particle from the origin to a point $P(x_p,y_p)$ in the $n$'th frame is given by

$$r_n^2 = (x_n - x_0)^2 + (y_n - y_0)^2.$$ 

where $n$ runs from first to last frame. The mean square displacement of all particles in the $n$'th frame is

$$\langle r_n^2 \rangle = \frac{\sum_{p=1}^{N} r_n^2}{N}.$$ 

The final task is to plot $\langle r_n^2 \rangle$ for each frame against time.

The image processing, particle locating, centroid finding and particle tracking are shown as a sequence of steps in the flow chart of Figure 11.8.

**Q 5.** Plot the average $\langle r_n^2 \rangle$ against time.
Q 6. Plot $r_2^2$ against time for a representative microparticle.

Q 7. Plot a graph of Brownian motion exhibited by the particles in the 2-dimensional xy-plane. Does the plot look like the random motion of actual particles in the movie?

**Calculation of Boltzmann constant $k_B$ and Diffusion constant $D$**

Boltzmann’s constant $k_B$ is calculated using the slope of the $\langle r^2 \rangle$ vs time curve. The slope value after conversion of units of displacement from pixel$^2$ to $\mu$m$^2$ is substituted into Equation (11.8). The diffusion constant is calculated using Equation (11.9).

Q 8. Calculate $k_B$ and $D$ for your data. Refer to [15] to find the value of viscosity at the temperature of laboratory.

Q 9. Find the uncertainty in the value of $D$.

Q 10. Calculate Avogadro’s number $N_A$ using the value of slope and Equation (11.10). What is the uncertainty in $N_A$?

### 11.5 Precautions in microscope handling

1. The light source in the microscope which is a halogen bulb is provided with the intensity control knob. Always try to focus the objective starting with the lowest values by rotating the knob clockwise. Never turn the knob abruptly to its extreme position of maximum intensity. This can damage the objective lens of the microscope.

2. While switching the viewing mode from eyepiece to camera using the optical path changeover lever, always minimize the intensity first.

3. Always switch off the microscope at the least intensity value i.e., zero.

4. Never rotate the left and right Coarse or Fine knob while holding the other.

5. Never turn the Coarse and Fine focus knobs further than their limit.
Chapter 12

The Magnetic Pendulum

Junaid Alam and Muhammad Sabieh Anwar

Nonlinearity is a profound concept in the study of physical systems. The characteristics of seemingly very simple systems may turn out to be extremely intricate and practically unpredictable due to the nonlinearities associated with them. The study of chaos—that essentially has its basis in nonlinear dynamics—begins with the study of such simple systems. The magnetic pendulum can be one such system.

A pendulum is one of the most simple and affluent systems in terms of its mathematical basis and the range of fields of science that it can relate to. Without a doubt, it is a gift of reflective simplicity for our reductionist science. With slight modifications, it can exhibit even more insightful phenomena, chaos being one of them. In this experiment, we will explore the notion of chaotic dynamics using a "magnetic pendulum".

KEYWORDS

Determinism · Chaos · Supersensitivity · Phase Portrait · Poincare Map · Attractor · Fractals · Self-similarity · Resonance · Rotary motion sensor.

PREREQUISITE EXPERIMENT: Chasing Chaos with an RL-Diode Circuit

APPROXIMATE PERFORMANCE TIME: 1 week
12.1 Objectives

In this experiment, we will discover:

1. how apparently simple systems can be highly non-linear and exhibit a complex behavior under certain conditions,

2. how non-linearity can be made more prominent using simple methods and hardware modifications,

3. simplistic richness of the mathematical and physical structure of dynamical systems,

4. the conditions and consequences of the notion of super-sensitivity and its relationship with chaos,

5. how to tell chaos apart from statistical indeterminacy,

6. how to construct and interpret phase portraits and Poincare Maps for different kinds of responses of a system,

7. how fractals are associated with attractors and are manifest in the graphical data of such systems.
Bibliography


12.2 Introduction

The Basics: Write the equation of motion for a simple pendulum and identify the terms that can represent the associated nonlinearity.

Linearize: Under what conditions, a pendulum can be treated as a linear system?

Almost all of the known physical systems are essentially nonlinear. Yet, for simplicity, they can be treated as linear systems within some operating constraints.
The magnetic pendulum is one such system: although it is treated as a linear device in elementary mechanics, when treated accurately, it is a nonlinear system. So, it can help us to look into some nonlinear phenomena like chaos.

12.3 Apparatus

The magnetic pendulum capable of exhibiting chaotic dynamics is required to fulfill the following demands:

- variable amplitude and frequency of the driving force,
- variable natural frequency of the oscillator and the extent of nonlinearity,
- adjustable damping force mechanism, and
- a way to display the graphical data obtained from the system (such as time series, phase plots, Fourier spectra etc.)

We built one such apparatus whose schematic is shown in Figure 1.

**The math ingredient:** The setup shows a pendulum being driven by a horizontally oscillating pivot. Derive the equation of motion for such a pendulum.

Setup Description

Some important parts of the setup are briefly described below.

**AC Induction Motor**

An AC induction motor has been used as the driving device. We use it for its readily available power supply (without any drive needed) and for its ease in speed control, which is simply achieved using a fan-dimmer.

**Flywheel and Connecting Rod**

The flywheel and connecting rod assembly converts the circular motion of motor’s shaft to a linear simple harmonic motion of the bearing-rod assembly to which the pendulum and rotary motion sensor are attached.
Figure 12.1: The magnetic pendulum apparatus schematic

**Think:** What is the advantage of using a flywheel?

**Rails and Bearing-Rod-Assembly**

A rod with two bearings fitted on it is placed in rails, so as to support the pendulum as well as to keep the rod from undergoing any non-lateral movement.

**Ask Yourself:** Was it necessary to break the connecting rod into two?

**Rotary Motion Sensor**

Vernier rotary motion sensor (RMS) encodes the angular information of the shaft into a digital stream and sends it to your LabVIEW program through LabPro. This information can then be used in MATLAB for further processing.
Disc-and-Magnet damping mechanism

A lightweight aluminum disc is integrated with the rod on which the pendulum is suspended. This provides us with magnetic damping (or eddy current damping) when the magnet (attached to the side of RMS) is brought near the disc.

Ring Magnet

Large ring magnet provides a magnetic field to interact with that of the smaller magnet at the end of pendulum. In this way, we can control the magnitude and nature of the restoring torque and hence the nonlinearity of pendulum.

Differentiate: What difference will it make if we use a small magnet similar to the one attached to pendulum instead of large ring magnet?

Design idea: Can you design a better setup to meet the same qualitative requirements?

12.4 The Experiment

The experiment is divided into three parts:
1. Exploring the nonlinearities

2. Driving the system into chaos

3. The graphical analysis

Exploring the Nonlinearities

Let us start our experimental expedition.

1. Measure the time period of pendulum in unforced and undamped mode using a stop watch, keeping the amplitude small.

2. Now, increasing the amplitude, measure the time period again and compare the measurements for several amplitudes.

3. Place the ring magnet under the pendulum and measure the time periods for different amplitudes and ring magnet orientations.

   Compare: What can you learn from the measurements of time period, with the ring magnet placed under the pendulum?

   Relate: How can you relate the results with the degree of nonlinearity of the system?

4. Run the file *RotaryMotion.vi* from your computer desktop. Collect the data for free oscillations for different scenarios of amplitude and nonlinearity.

5. Import the data into MATLAB and observe the time series, phase plots and Fourier spectra of collected data.

   Observe: How is the increased nonlinearity indicated in the graphical data? Can you identify a graph for higher degree of nonlinearity from smaller degrees of nonlinearity?

Stepping into Chaos

Precaution: Never run the motor at high speeds. It may damage the apparatus.
Start from very low speeds and gradually increase when needed. If the motor instantaneously gains speed, switch it off immediately.
1. Place the ring magnet at about a distance of 7 cm under the pendulum so that you can clearly observe the stable regions on both sides of the pendulum’s mean position.

2. Turn the motor dimmer anticlockwise to its ZERO and switch the motor ON. Now slowly turn the dimmer clockwise until the motor starts.

3. Measure the angular frequency of the drive using the smart timer and make any adjustments if needed. Keep the angular speed in the range of 1 - 2 cycles per second. Measure for both periodic and chaotic behaviors.

4. Once the speed is stabilized, run the LabVIEW file (RotaryMotion.vi) and collect the data for different drive frequencies and different spacings between the ring magnet and pendulum.

5. Import the data into MATLAB and plot the time series, phase plots and Fourier Spectra of the data. Have them saved in a folder named as your roll number and get them printed to attach in the notebooks.

   **Juxtapose:** How are the periodic and chaotic responses different in the graphical representation? How a more nonlinear periodic response differs from a lesser one?

   **Poincare Sections**

6. Drive the system to exhibit periodic response. By putting the sample time equal to the drive frequency, collect the data and plot it as a phase plot.

7. Now, driving the system into chaos, and again setting the sample time equal to the drive cycle, plot the data in the phase space.

   **Contrast:** Observe the difference in the Poincare Maps for the two cases and explain its underlying concept briefly.

**The Graphical Analysis**

The purpose of this section is to get acquainted with the graphical richness of chaos and the notion of attractors.

**Point Out** How graphical data helps in determining if the system is chaotic or not?

**Dive deeper:** What should a chaotic phase portrait should look like?
Make Out: Can you identify the attractors, repellers or saddle points in the phase portrait, if any?

Magnify: Can you find a connection between attractors and fractals from the graphical data?
Chapter 13

Synthesis and Ferroelectric Properties of KNO$_3$ films

Shahid Ramay and Muhammad Sabiheh Anwar

Potassium nitrate (KNO$_3$) shows ferroelectric behavior at temperatures between $-152^\circ$C and $120^\circ$C due to change in its crystal structure and phase. In the current experiment, we will study the ferroelectric behavior of this material as a function of temperature with a homemade Sawyer Tower circuit. Through this experiment, we will explore phase transitions, and will appreciate how these transition lead to different physical properties. The take home lesson is that these properties are intimately connected to the underlying crystal structure.

**Keywords** Ferroelectricity, phase transition, coercive field, spontaneous electric polarization, hysteresis

**Approximate Performance Time:** One week
Bibliography


13.1 List of Equipment

1. Dual channel oscilloscope

2. Variac

3. Microwave oven

4. Hot plate up to 400°C

5. Thermocouple

6. Al sheet
7. KNO₃ powder
8. Iron oxide thin films (optional, for investigating ferroelectric behavior of nanosized thin films)
9. High wattage resistors
10. High voltage capacitor
11. Thumb pins
12. Insulated gloves
13. Connecting wires
14. Crocodile clips
15. Glass microscope slides

13.2 Objectives

In this experiment students will learn,

1. to make a thin layer of KNO₃ with the melt technique
2. how moisture affects the ferroelectric properties of KNO₃.
3. how temperatures affects the coercive field and spontaneous polarization of KNO₃, and
4. the role of crystal structure in determining the material properties.

13.3 Theoretical background

The ferroelectric effect was first discovered by Valasek in 1921, in Rochelle salt, KNaC₂H₆O₆·H₂O.

Electric dipole moment

A ferroelectric material must possesses a spontaneous electric dipole moment that can be switched in an applied electric field. This effect is found when two particles of charge q are separated by a distance r.

\[ \vec{D} = q \vec{r}. \quad (13.1) \]
Here $\mathbf{r}$ is the dipole moment, $q$ is the charge and $r$ is the vectorial distance between the two charges.

**Electric polarization**

All solids consist of charged particles (nuclei and electrons) but overall they are neutral. For most solids, there is also no net separation of positive and negative charges; there is no net dipole moment. Even if a solid is composed of molecules with permanent dipole moments (e.g. ice), the molecules are generally arranged in such a way that the unit cell of the crystal and the bulk solid have no net dipole moment. If this solid is placed in an electric field then a field is induced in the solid which opposes the applied field. This field arises from two sources:

1. a distortion of the electron cloud of the atoms or molecules, and
2. slight movement of the atoms themselves.

The average dipole moment per unit volume induced in the solid is called the electrical polarization and is denoted as $P$. Ferroelectric materials are preferably polarized in certain crystallographic directions.

**Ferroelectricity and crystal structure**

Ferroelectricity is exhibit only in materials with a specific crystal structure. Ferroelectricity does not exist in centrosymmetric materials because any dipole moment generated in one direction would be forced by symmetry to be zero. Besides non-centrosymmetry, there must also be a spontaneous local dipole moment. This means that central atom must be in a non-equilibrium position, leading to a non-overlap of the centers of gravity of the positive and negative charges. This concept is illustrated in Figure 13.1.

**Ferroelectricity in BaTiO$_3$**

Barium titanate BaTiO$_3$ is a classic example used for demonstration the role of crystal structure in determining the ferroelectric properties. BaTiO$_3$ is a ferroelectric material having very large dielectric constant ($\gtrsim 1000$) and is widely used in making ceramic capacitors. This material has the following phases with each phase being a unique crystal structure.
Figure 13.1: In (A) the structure is said to be non-polar. There is no displacement of the central atom, and no net dipole moment. In (B), however, the central atom is displaced and the structure is polar, possessing a spontaneous electric dipole moment.

Figure 13.2: (a) Shows the perovskite structure of BaTiO₃ with the TiO₆ environment depicted for one of the corner Ti⁴⁺ ion. (b) shows a space filling model of the perovskite structure: the TiO₆ octahedra fill up the volume, sharing their corners with one another. In this model, the large Ba²⁺ ions snugly fit into the interstices between the octahedra.

**Cubic BaTiO₃**

Above 120°C, BaTiO₃ has cubic crystal lattice. The unit cell comprises a Ba²⁺ ion in the center, Ti⁴⁺ ions at the cube corners and O²⁻ ions at the centres of the cube edges. The structure, called a perovskite structure, is shown in Figure 13.2. Several important ceramics and high-temperature superconductors possess perovskite structures. In the language of crystallography, the fractional coordinates of the ions are Ti: (0,0,0), Ba: (1/2,1/2,1/2) and O: (1/2,0,0),(0,1/2,0),(0,0,1/2). If you observe carefully, each Ti ion is in fact
surrounded by a octahedron of O\textsuperscript{2–} ions. Since the ionic radius of Ti\textsuperscript{4+} is very small (75 pm) as compared to oxygen, there is plenty of room for it to move inside the oxygen cage. The cubic phase is non-ferroelectric.

Q 1. The Ti-O bond length is 1.953 Å. What is the Ti-Ba bond length? In the cubic crystal system, all edges of the unit cell are of equal length.

Q 2. Why is cubic BaTiO\textsubscript{3} non-ferroelectric?

**Tetragonal BaTiO\textsubscript{3}**

As the material is cooled, the cubic lattice undergoes transformations. For example, it changes into tetragonal at 120°C. In the tetragonal phase, the cube distorts. As a result the octahedral TiO\textsubscript{6} group also distorts and the Ti\textsuperscript{4+} ion displaces along a Ti-O bond axis. The displacement of the Ti\textsuperscript{4+} ion causes a non-overlap of the positive and negative charge centres, resulting in a permanent electric dipole moment, a concept that is illustrated in Figure 13.3, conferring ferroelectric properties to the structure. The ion can be off-center in six possible directions (six O’s surround an individual Ti). Due to these possibilities, neighboring domains have electric polarizations that are either 90 or 180 degrees with respect to each other. The possibilities of dipole moments that are 180 and 90 degrees with respect to each other are illustrated in Figure 13.4.

Figure 13.3: Displacement of a Ti\textsuperscript{4+} ion within the octahedral oxygen cage, conferring ferroelectric properties.
Figure 13.4: I and II show electric dipoles that are 180° with respect to each other whereas I and III depict dipoles that are aligned at 90° with respect to each other. (This figure is reproduced from [7].)

**Orthorhombic or Rhombohedral BaTiO$_3$**

Upon further cooling of BaTiO$_3$, Ti$^{4+}$ starts to move along a diagonal between two Ti-O bonds, and at −90°C, a complete rhombohedral or orthorhombic ferroelectric phase is formed.

**Ferroelectric hysteresis**

Polarization can be reversed by applying a large alternating field and this produces a hysteresis loop between the electric field $E$ and polarization $P$. The polarization and electric displacement $D$ become non-linear functions of the electric field. They are related to each other through the constitutive relationship,

$$D = P + \varepsilon E.$$  (13.2)
13.3. **THEORETICAL BACKGROUND**

Usually in ferroelectric materials, the second term is negligible and a $D-E$ becomes interchangeable with a $P-E$ loop. From the saturated $P-E$ loop, we can also measure coercive field ($E_c$) and remanent polarization $P_r$. A ferroelectric hysteresis loop is both frequency and temperature dependent and a loop is shown in Figure 13.5.

![Hysteresis Loop](image)

**Figure 13.5:** Hysteresis loop of a typical ferroelectric material.

**Q 3.** What are the differences and similarities between (a) dielectric and ferroelectric, (b) ferroelectric and ferromagnetic materials?

**Measurement of polarization with a Sawyer-Tower circuit**

Polarization can be measured with the help of a standard Sawyer-Tower circuit, shown in Figure 13.6. An a.c. field applied across the sample is attenuated by a resistive divider, and the current is integrated into charge by virtue of a large capacitor $C_{\text{ref}}$ in series with the sample. Both these voltages are fed into the X and Y channels of an oscilloscope operating in the dual mode to generate the $P-E$ loop. In our case, we use high wattage resistors $R_1 = R_2 = 100$ and a reference capacitor of $C_{\text{ref}} = 0.4 \, \mu F$ with a high voltage rating. The applied a.c. voltage is 25 V.

**Q 4.** Based on simple circuit analysis arguments, explain the working of the Sawyer-Tower circuit with the assumption $C_{\text{ref}} > C_{\text{sample}}$?
13.4 The Experiment

In the present experiment, we will investigate the ferroelectric behavior of KNO₃ instead of BaTiO₃. Even though KNO₃ has a rather complicated structure [7], but its film is easier to synthesize in the short time period allocated for the present experiment. The description of BaTiO₃ serves as a useful reminder that the crystal structure determines the ferroelectric properties of the material.

BaTiO₃ has a melting point of 1650°C while KNO₃ melts at 330°C. It is easy to make thin film of KNO₃ by melting as compared to BaTiO₃. Be warned that KNO₃ is a strong hydrophilic and water kills the ferroelectricity of the material, so before use the nitrate must be dried in a conventional microwave oven for about two minutes. KNO₃ is highly conductive in the molten state, so extreme care must be taken when it is dried in the microwave oven. Do not overheat, as arcing may occur in the molten conductive state of the salt.

KNO₃ is applied on the surface of an Al sheet. The KNO₃ is melted while the Al sheet is clamped to a hotplate. Inverted thumbpins are placed on the KNO₃ layer. Upon melting the thumbpins automatically adhere to the KNO₃ form a capacitor of some thickness. At 330°C, KNO₃ is fully melted and the heater is turned off. The thermometer is being constantly monitored by a multimeter equipped with a thermocouple.

Now apply a 25 volts amplitude a.c. signal with the help of the variac, as the coercive voltage is about 17-20 volts, the precise depending on the thickness of the sample. As the capacitor goes through the phase transition, the loop will change dramatically and become very square.
Figure 13.7: (a) Heat KNO₃ powder in a conventional microwave oven, then (b) melt KNO₃ powder on Al sheet to make thin film with inverted thumbpins acting as electrodes.

Q 5. Record temperature dependent hysteresis loops for your KNO₃ films.

Q 6. Replace your sample with a conventional capacitor. What do you observe on the oscilloscope screen and why?