1. The eigenfunctions for a potential energy function

\[ V(x) = \begin{cases} \infty & a < x < 0 \\ 0 & 0 < x < a, \end{cases} \]

are of the form

\[ \psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a}. \]

Suppose a particle in this potential has an initial normalized wave function of the form

\[ \Psi(x, 0) = A \left( \cos \left( \frac{\pi x}{a} \right) \right). \]

What is \( \Psi(x, t) \)?

**Answer**

We have to find the time-dependent wavefunction \( \Psi(x, t) \);

\[
\Psi(x, t) = \langle x | \psi(t) \rangle = \langle x | e^{-i\hat{H}t/\hbar} | \psi(0) \rangle = \langle x | e^{-i\hat{H}t/\hbar} \sum_n c_n(0) | \psi_n \rangle = \sum_n \langle x | c_n(0) e^{-iE_n t/\hbar} | \psi_n \rangle = \sum_n c_n(0) e^{-iE_n t/\hbar} \psi_n(x),
\]

where \( \psi_n(x) = \sqrt{\frac{2}{a}} \sin \left( \frac{n\pi x}{a} \right) \) are the eigenstates of the Hamiltonian and

\[
c_n(0) = \langle \psi_n | \Psi(x, 0) \rangle = \int_0^a dx \langle \psi_n | x \rangle \langle x | \Psi(x, 0) \rangle = \int_0^a dx \psi_n^\dagger(x) \Psi(x, 0)
= \int_0^a dx \sqrt{\frac{2}{a}} \sin \left( \frac{n\pi x}{a} \right) A \cos \left( \frac{\pi x}{a} \right)
= \sqrt{\frac{2}{a}} A \int_0^a dx \sin \left( \frac{n\pi x}{a} \right) \cos \left( \frac{\pi x}{a} \right). \tag{1}
\]
Now
\[ \int_0^a dx \sin \left( \frac{n\pi x}{a} \right) \cos \left( \frac{\pi x}{a} \right) = \frac{1}{2} \int_0^a dx \sin \left( (n + 1) \frac{\pi x}{a} \right) + \sin \left( (n - 1) \frac{\pi x}{a} \right) \]
\[ = \frac{1}{2} \left[ \cos(n + 1) \right] - \cos(n - 1) \left[ \frac{\pi a}{(n + 1) \pi} \right] \]
\[ = \frac{a}{2\pi} \left[ \frac{1 - \cos(n + 1)\pi}{n + 1} + \frac{1 - \cos(n - 1)\pi}{n - 1} \right] \]
\[ = \frac{a}{2\pi} \left[ (1 + \cos(n\pi)) \left( \frac{1}{n + 1} + \frac{1}{n - 1} \right) \right] \]
\[ = \frac{a}{2\pi} \left[ (1 + \cos(n\pi)) \left( \frac{2n}{n^2 - 1} \right) \right] = \frac{a n}{\pi(n^2 - 1)} \left( 1 + \cos(n\pi) \right) \].

Inserting this expression into Eq. (1),
\[ c_n(0) = \sqrt{\frac{2}{a}} A \left( \frac{a n}{\pi(n^2 - 1)} \right) \]
\[ \Rightarrow \Psi(x, t) = \sum_n \left( \sqrt{\frac{2}{a}} A \left( \frac{a n}{\pi(n^2 - 1)} \right) \right) e^{-iE_n t/\hbar} \sqrt{\frac{2}{a}} \sin \left( \frac{n\pi x}{a} \right) \]
\[ = \left( \frac{2A}{\pi} \right) \sum_n e^{-iE_n t/\hbar} \frac{n}{(n^2 - 1)} \left( 1 + \cos(n\pi) \right) \sin \left( \frac{n\pi x}{a} \right) \]
\[ = \left( \frac{2A}{\pi} \right) \sum_{n=\text{even}} e^{-iE_n t/\hbar} \frac{2n}{(n^2 - 1)} \sin \left( \frac{n\pi x}{a} \right) \text{ (Terms with odd } n \text{ vanish)} \]

2. You are given the Gaussian wavepacket at zero time, representing an electron,
\[ \Psi(x, 0) = \frac{1}{(\sigma \sqrt{\pi})^{1/2}} e^{-x^2/2\sigma^2} e^{i\phi} \]

(a) What is the value of \( \langle p \rangle \) for this quantum state at time \( t = 0 \)?

Use the integral
\[ \int_{-\infty}^{\infty} dy e^{-ay^2} = \sqrt{\frac{\pi}{a}}. \]

(b) What is the uncertainty in momentum of this state. Use
\[ \int_{-\infty}^{\infty} dy y^2 e^{-ay^2} = \frac{1}{2} \sqrt{\frac{\pi}{a^3}}. \]
Answer

(a) The easiest way to find the expectation value of momentum at time $t = 0$, is through wavefunction in position space,

$$\langle p \rangle = \langle \Psi(x, 0) | \hat{p} | \Psi(x, 0) \rangle$$

$$= \langle \Psi(x, 0) | -i\hbar \frac{\partial}{\partial x} | \Psi(x, 0) \rangle$$

$$= \int_{-\infty}^{\infty} dx \left( \frac{1}{(\sigma \sqrt{\pi})^{1/2}} e^{-\left(\frac{x^2}{2\sigma^2}\right)} e^{-i\hbar p_0 x / \hbar} \right) \left( -i\hbar \frac{\partial}{\partial x} \right) \left( \frac{1}{(\sigma \sqrt{\pi})^{1/2}} e^{-\left(\frac{x^2}{2\sigma^2}\right)} e^{i\hbar p_0 x / \hbar} \right)$$

$$= -i\hbar \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2\sigma^2}} -i\hbar \frac{\partial}{\partial x} \left( e^{-\frac{x^2}{2\sigma^2} + \frac{i\hbar p_0 x}{\hbar}} \right)$$

$$= -i\hbar \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2\sigma^2}} \left( \frac{-x}{\sigma^2} + \frac{i\hbar p_0}{\hbar} \right)$$

$$= -i\hbar \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2\sigma^2} + \frac{i\hbar p_0}{\hbar}} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2\sigma^2}}$$

$$= -i\hbar \sqrt{\frac{\pi}{\sigma}} \int_{-\infty}^{\infty} dx \left[ 0 + \frac{i\hbar p_0}{\hbar} \sqrt{\frac{\pi}{\sigma}} \right] = p_0$$

to use $\int_{-\infty}^{\infty} dy e^{-ay^2} = \sqrt{\frac{\pi}{a}}$

(b) To find the uncertainty in momentum $\Delta p$, first we need to find the expectation value of $p^2$;

$$\langle p^2 \rangle = \langle \Psi(x, 0) | \hat{p}^2 | \Psi(x, 0) \rangle$$

$$= \frac{1}{\sigma \sqrt{\pi}} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2\sigma^2}} \cdot \frac{\partial^2}{\partial x^2} \left( e^{-\frac{x^2}{2\sigma^2} + \frac{i\hbar p_0 x}{\hbar}} \right)$$

$$= -\frac{\hbar^2}{\sigma \sqrt{\pi}} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2\sigma^2}} \cdot \frac{\partial}{\partial x} \left[ \left( \frac{-x}{\sigma^2} + \frac{i\hbar p_0}{\hbar} \right) e^{-\frac{x^2}{2\sigma^2} + \frac{i\hbar p_0 x}{\hbar}} \right]$$

$$= -\frac{\hbar^2}{\sigma \sqrt{\pi}} \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2\sigma^2}} \cdot \left[ \frac{-1}{\sigma^2} + \left( \frac{-x}{\sigma^2} + \frac{i\hbar p_0}{\hbar} \right) \right] e^{-\frac{x^2}{2\sigma^2} + \frac{i\hbar p_0 x}{\hbar}}$$

$$= -\frac{\hbar^2}{\sigma \sqrt{\pi}} \left[ \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2\sigma^2}} + \int_{-\infty}^{\infty} dx e^{-\frac{x^2}{2\sigma^2}} \left( \frac{-x}{\sigma^2} + \frac{i\hbar p_0}{\hbar} \right) \right]$$

$$= -\frac{\hbar^2}{\sigma \sqrt{\pi}} \left[ \frac{-1}{\sigma^2} \sqrt{\pi \sigma^2} + \frac{1}{\sigma^4} \int_{-\infty}^{\infty} dx \cdot x e^{-\frac{x^2}{2\sigma^2}} - \frac{p_0^2}{\hbar^2} \int_{-\infty}^{\infty} dx \cdot x e^{-\frac{x^2}{2\sigma^2}} - \left( \frac{2i\hbar p_0}{\sigma^2 \hbar} \right) \int_{-\infty}^{\infty} dx \cdot x e^{-\frac{x^2}{2\sigma^2}} \right]$$

$$= -\frac{\hbar^2}{\sigma \sqrt{\pi}} \left[ \frac{-1}{\sigma} + \frac{1}{\sigma^4} \left( \frac{1}{\sigma^4} \frac{p_0^2}{\hbar^2} \frac{\sqrt{\pi \sigma^2}}{2\sigma} - \frac{p_0^2 \sqrt{\pi}}{\hbar^2} \right) \right] - \frac{\hbar^2}{\sigma}$$

$$= -\frac{\hbar^2}{\sigma} \left[ \frac{-1}{\sigma} + \frac{\sqrt{\pi \sigma^2}}{2\sigma} - \frac{p_0^2 \sqrt{\pi}}{\hbar^2} \right] = \frac{\hbar^2}{\sigma^2} - \frac{\hbar^2}{2\sigma^2} + p_0^2 = \frac{\hbar^2}{2\sigma^2} + p_0^2.$$
Therefore,

\[(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2\]

\[= \frac{\hbar^2}{2\sigma^2} + p_0^2 - p_0^2 = \frac{\hbar^2}{2\sigma^2},\]

and the uncertainty is \(\Delta p = \sqrt{(\Delta p)^2} \sim \hbar/\sqrt{2}\sigma\).