1. A polarizer $P$ has its transmission at an angle $\phi$ with respect to the vertical, as shown in Fig. (1). A beam of photons prepared in the state $|V\rangle$ enters the polarizer.

(a) What kind of measurement does the polarizer perform? It is modeled as a projection operator. Write this operator in the bra-ket notation. [4]

(b) What is the output state of the photons leaving the polarizer? [3]

(c) If $N$ photons are incident on $P$, how many are transmitted on average? [3]

(d) In Fig. (2), $P_2$ has a horizontal transmission axis. What fraction of photons are transmitted from $P_2$? [2]

(e) Now we have a set of cascaded polarizers. The transmission axis successively differs by $\phi/M$. Therefore, $P_1$ has its axis at $\phi/M$, $P_2$ at $2\phi/M$ and so on. The arrangement is shown in Fig. (3) overleaf. There are $M$ polarizers in total. Using your result in (c), what fraction of photons are transmitted from $P_M$? What fraction are transmitted if $M >> \phi$? If $\phi = \pi/2$ and $M$ is large why isn’t the output state annihilated as naively
expected from the result in (d). [6]
(Hint: Use $\cos \theta \approx 1 - \frac{\theta^2}{2!} + \ldots$ to the lowest order).

![Diagram](image)

**Answer**

(a) The projection operator in bra and ket notation is given by

$$\hat{P}_\alpha = |\alpha\rangle \langle \alpha|,$$

where $|\alpha\rangle$ state for the polarizer $P$ having its transmission at an angle $\phi$ with respect to the vertical becomes

$$|\alpha\rangle = \cos \left( \frac{\pi}{2} - \phi \right) |H\rangle + \sin \left( \frac{\pi}{2} - \phi \right) |V\rangle = \sin \phi |H\rangle + \cos \phi |V\rangle$$

$$\hat{P}_\alpha = \left( \sin \phi |H\rangle + \cos \phi |V\rangle \right) \left( \sin \phi \langle H| + \cos \phi \langle V| \right)$$

$$= \sin^2 \phi |H\rangle \langle H| + \cos^2 \phi |V\rangle \langle V| + \sin \phi \cos \phi |H\rangle \langle V| + \cos \phi \sin \phi |V\rangle \langle H|.$$}

The polarizer is a projection operator $\hat{P}_\alpha$. It is Hermitian ($\hat{P}_\alpha = \hat{P}_\alpha^\dagger$). The observable is whether the photon is transmitted or not. If the photon is transmitted, it will be in the state $|\alpha\rangle$ which is an eigenstate of $\hat{P}_\alpha$. The probability of transmission is $|\langle \alpha|\hat{P}_\alpha|\psi_{in}\rangle|^2$. The probability of blockage is $(1 - |\langle \alpha|\hat{P}_\alpha|\psi_{in}\rangle|^2)$. The measurement outcomes (eigenvalues) are 1 and 0 for transmission and no transmission respectively.

(b) The output state $|\psi_{out}\rangle$ will be proportional to $|\alpha\rangle = \sin \phi |H\rangle + \cos \phi |V\rangle$. This can

Date: Mar. 3, 2014
be shown as below

\[ |\psi_{\text{out}}\rangle = \hat{P}_a|\psi_{\text{in}}\rangle = \hat{P}_a|V\rangle \]

\[ = \left( \sin^2 \phi |H\rangle \langle H| + \cos^2 \phi |V\rangle \langle V| + \sin \phi \cos \phi |H\rangle \langle V| + \cos \phi \sin \phi |V\rangle \langle H| \right) |V\rangle \]

\[ = \cos^2 \phi |V\rangle + \sin \phi \cos \phi |H\rangle \]

\[ = \cos \phi \left( \cos \phi |V\rangle + \sin \phi |H\rangle \right). \]

(c) The fraction of photons transmitted will be \( |\langle \psi_{\text{out}}|\psi_{\text{out}}\rangle|^2 \) given that the input state is normalized, \( \langle \psi_{\text{in}}|\psi_{\text{in}}\rangle = 1 \). Now we need to find \( \langle \psi_{\text{out}}|\psi_{\text{out}}\rangle \):

\[ \langle \psi_{\text{out}}|\psi_{\text{out}}\rangle = \left( \cos \phi \left( \cos \phi |V\rangle + \sin \phi |H\rangle \right) \right) \left( \cos \phi \left( \cos \phi |V\rangle + \sin \phi |H\rangle \right) \right) \]

\[ = \cos \phi \left( \cos^2 \phi \langle V|V\rangle + \sin^2 \phi \langle H|H\rangle \right) \]

\[ = \cos \phi \left( \cos^2 \phi + \sin^2 \phi \right) = \cos \phi. \]

So the fraction will be \( \cos^2 \phi \).

(d) Zero. \( P_1 \) prepares photons in the state \( |V\rangle \) with a probability one, which are completely blocked by the horizontal polarizer, i.e.,

\[ P_2|V\rangle = |H\rangle \langle H|V\rangle = 0. \]

This can also be seen by plugging \( \phi = \pi/2 \) in result of part (c): \( \cos^2(\pi/2) = 0 \). So no photons are transmitted from \( P_2 \).

(e) (i) As \( P_1 \) has its axis at \( \phi/M \), so the fraction transmitted from \( P_1 \) is \( \cos^2(\phi/M) \). For the given arrangement of cascaded polarizers, each successive polarizer transmits \( \cos^2(\phi/M) \) of photons incident on it. Hence total transmittivity is

\[ \left( \cos^2(\phi/M) \right)^M = \left( \cos(\phi/M) \right)^{2M}. \]

(ii) If \( M \) is large \( \phi/M << 1 \),

\[ \cos(\phi/M) \approx 1, \quad \therefore \cos \theta \approx 1 \quad \text{for small } \theta \]
then

\[(\cos \phi / M)^{2M} \approx 1.\]

All the photons will be transmitted.

(iii) The result is different from what we expected from part (d). Here each polarizer is performing a measurement, collapsing the state into \( |\pi/2 - k\phi \rangle_M \) where \( k \) is an integer. The next polarizer is oriented approximately parallel to the plane of polarization of the photon, transmitting the photon. Hence the photon, to first order, suffers no blockage and is transmitted with almost complete certainty. This is called the quantum Zeno effect. Placing just one polarizer at \( \phi = \pi/2 \) completely blocks the \( |V\rangle \) beam because no intervening measurement is performed.

2. (a) Show that if \( |i\rangle \)'s form an orthonormal basis, with \( i = 1, 2, 3, \ldots, N \), then the \( \hat{U}|i\rangle \)'s also form a basis, provided \( \hat{U} \) is a unitary operator. [3]

(b) Show that the trace of a Hermitian operator is equal to the sum of its eigenvalues. [3].

**Answer**

(a) Since \( |i\rangle \)'s states form an orthonormal basis, then

\[
\sum_{i=1}^{N} |i\rangle \langle i| = \hat{1}.
\]

Let \( \hat{U}|i\rangle = |b_i\rangle \). If the \( |b_i\rangle \)'s states are to form a basis, we must have

\[
\sum_{i=1}^{N} |b_i\rangle \langle b_i| = \hat{1},
\]

where L.H.S is

\[
\sum_{i=1}^{N} \hat{U}|i\rangle \langle i|\hat{U}^\dagger = \hat{U} \left( \sum_{i=1}^{N} |i\rangle \langle i| \right) \hat{U}^\dagger = \hat{U} \hat{1} \hat{U}^\dagger = \hat{U} \hat{U}^\dagger.
\]

But this is \( \hat{1} \) only if \( \hat{U}^{-1} = \hat{U}^\dagger \) showing that \( \hat{U} \) must be unitary.

(b) Let \( \hat{A} \) is an Hermitian operator, the eigenvalue corresponding to its eigenstate \( |\lambda_i\rangle \) is \( \lambda_i \),

\[
\hat{A}|\lambda_i\rangle = \lambda_i|\lambda_i\rangle.
\]
We define
\[ \hat{A}_{ij} \equiv \langle \lambda_i | \hat{A} | \lambda_j \rangle. \]

The trace \( \text{Tr}(\hat{A}) \) of an operator \( \hat{A} \) is given, within an orthonormal basis \( \{ |\lambda_i\rangle \} \), by the expression
\[
\text{Tr}(\hat{A}) = \sum_i \langle \lambda_i | \hat{A} | \lambda_i \rangle = \sum_i \hat{A}_{ii}
\]
\[ = \sum_i \lambda_i \langle \lambda_i | \lambda_i \rangle \quad \because \hat{A} \text{ is Hermitian}
\]
\[ = \sum_i \lambda_i, \]
which is the required condition.

3. (a) A polarization rotator is an optical element that rotates the plane of polarization of a photon through an angle \( \alpha \). Represent this operator in the \( \{ |H\rangle, |V\rangle \} \) basis. [3]

(b) Express this operator in a basis that is rotated at an angle \( \pi/4 \) with respect to the \( \{ |H\rangle, |V\rangle \} \) basis. First construct a similarity matrix. Use \( \hat{O} = \hat{S} \hat{O} \hat{S}^\dagger \). [6]

(c) Show that \( |L\rangle \) is an eigenstate of the rotation operator. [3]

(d) Consider Fig. (4).

The state \( |L\rangle \) is input to a polarizing beam splitter whose role is to separate the beam of photons to \( |H\rangle \) and \( |V\rangle \) channels. The rotator element \( \hat{O} \) is placed exclusively in the path of the \( |V\rangle \) photons. The second PBS\(_2\) combines the paths into one while the terminal circular polarization analyzer directs the photons to \( |L\rangle \) and \( |R\rangle \) channels. Finally \( D_L \) and \( D_R \) are demolitive detectors.

(i) Find the state \( |\psi_2\rangle \). [4]
(ii) What is the probability that $D_L$ clicks? Plot this probability as a function of $\alpha$.

(iii) If I block one beam, say the $|H\rangle$ photons in between the beamsplitters, what is the probability of $D_L$ clicking? Does it depend on $\alpha$? Your answer should convince you that interference is taking place between $|H\rangle$ and $|V\rangle$ possibilities.

(iv) If I remove the rotator $\hat{O}$, what is the probability that $D_L$ and $D_R$ click?

\textbf{Answer}

(a) Rotation of a horizontally polarized photon, yields the state

$$\hat{O}|H\rangle = \cos \alpha |H\rangle + \sin \alpha |V\rangle,$$

and rotation of a vertically polarized photon, yields

$$\hat{O}|V\rangle = -\sin \alpha |H\rangle + \cos \alpha |V\rangle.$$

The matrix representation of rotation operator in $\{|H\rangle, |V\rangle\}$ basis is thus

$$\hat{O} = \begin{pmatrix} \langle H|\hat{O}|H\rangle & \langle H|\hat{O}|V\rangle \\ \langle V|\hat{O}|H\rangle & \langle V|\hat{O}|V\rangle \end{pmatrix} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}.$$

(b) Suppose the new basis is represented as $\{|45^\circ\rangle, |135^\circ\rangle\}$ and a similarity matrix $\hat{S}$ in these basis is given by

$$\hat{S} = \begin{pmatrix} \langle 45^\circ|H\rangle & \langle 45^\circ|V\rangle \\ \langle 135^\circ|H\rangle & \langle 135^\circ|V\rangle \end{pmatrix} = \begin{pmatrix} \hat{S}_{11} & \hat{S}_{12} \\ \hat{S}_{21} & \hat{S}_{22} \end{pmatrix}.$$

Now

$$|45^\circ\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$$

$$|135^\circ\rangle = \frac{1}{\sqrt{2}}(-|H\rangle + |V\rangle).$$

So the matrix elements are

$$\hat{S}_{11} = \frac{1}{\sqrt{2}}, \quad \hat{S}_{12} = \frac{1}{\sqrt{2}},$$

$$\hat{S}_{21} = -\frac{1}{\sqrt{2}}, \quad \hat{S}_{22} = \frac{1}{\sqrt{2}}.$$
Hence the similarity matrix $\hat{S}$ is

$$\hat{S} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

The rotation operator in the new basis is given by,

$$\hat{O} = \hat{S} \hat{O} \hat{S}^\dagger$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \cos \alpha - \sin \alpha & -\cos \alpha - \sin \alpha \\ \sin \alpha + \cos \alpha & -\sin \alpha + \cos \alpha \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2 \cos \alpha & -2 \sin \alpha \\ 2 \sin \alpha & 2 \cos \alpha \end{pmatrix} = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix} = \hat{O}.$$

Hence the rotation operator $\hat{O}$ remains unchanged in the new basis.

(c) Now we have to show that $|L\rangle$ is an eigenstate of rotation operator $\hat{O}$:

$$\hat{O}|L\rangle = \hat{O} \frac{1}{\sqrt{2}}(|H\rangle+i|V\rangle)$$

$$= \frac{1}{\sqrt{2}} \left( \hat{O}|H\rangle + i\hat{O}|V\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left( \cos \alpha |H\rangle + \sin \alpha |V\rangle + i(\sin \alpha |H\rangle + \cos \alpha |V\rangle) \right)$$

$$= \frac{1}{\sqrt{2}} \left( (\cos \alpha - i \sin \alpha) |H\rangle + (\sin \alpha + i \cos \alpha) |V\rangle \right)$$

$$= \frac{1}{\sqrt{2}} \left( e^{-i\alpha} |H\rangle + ie^{-i\alpha} |V\rangle \right)$$

$$= e^{-i\alpha} \frac{1}{\sqrt{2}} (|H\rangle+i|V\rangle) = e^{-i\alpha} |L\rangle.$$

Hence $|L\rangle$ is an eigenstate with eigenvalue $e^{-i\alpha}$.

(d) (i) The input state $|\psi_{in}\rangle$ to a polarizing beam splitter PBS$_1$ is

$$|\psi_{in}\rangle = |L\rangle = \frac{1}{\sqrt{2}}(|H\rangle+i|V\rangle).$$

After the selective rotation operator $\hat{O}$, the output state from beam splitter PBS$_2$
becomes

\[ |\psi_2\rangle = \frac{1}{\sqrt{2}} (|H\rangle + i\hat{O}|V\rangle) \]
\[ = \frac{1}{\sqrt{2}} \left( |H\rangle + i(-\sin \alpha |H\rangle + \cos \alpha |V\rangle) \right) \]
\[ = \frac{1}{\sqrt{2}} \left( (1 - i \sin \alpha) |H\rangle + i \cos \alpha |V\rangle \right). \]

In bra and ket notation, the horizontal and vertical polarization states are

\[ |H\rangle = \frac{1}{\sqrt{2}} (|L\rangle + |R\rangle), \quad |V\rangle = \frac{i}{\sqrt{2}} (|R\rangle - |L\rangle). \]

So we have

\[ |\psi_2\rangle = \left( \frac{1}{\sqrt{2}} \right)^2 \left( (1 - i \sin \alpha) (|L\rangle + |R\rangle) - \cos \alpha (|R\rangle - |L\rangle) \right) \]
\[ = \frac{1}{2} \left( (1 - i \sin \alpha + \cos \alpha) |L\rangle + (1 - i \sin \alpha - \cos \alpha) |R\rangle \right). \]

(ii) The probability of \( D_L \) clicking is

\[ |\langle L|\psi_2\rangle|^2 = \frac{1}{4} (1 - i \sin \alpha + \cos \alpha)(1 + i \sin \alpha + \cos \alpha) \]
\[ = \frac{1}{4} ((1 + \cos \alpha) - i \sin \alpha)((1 + \cos \alpha) + i \sin \alpha) \]
\[ = \frac{1}{4} ((1 + \cos \alpha)^2 - (i \sin \alpha)^2) \]
\[ = \frac{1}{4} (1 + \cos^2 \alpha + 2 \cos \alpha + \sin^2 \alpha) \]
\[ = \frac{1}{2} (1 + \cos \alpha) = \cos^2 \left( \frac{\alpha}{2} \right). \]
(iii) If one beam $|H\rangle$ is blocked in between beam splitters, then the input state to PBS$_1$ is

$$|\psi_{\text{in}}\rangle = i|V\rangle.$$ 

Following the same procedure of part (i), after the selective rotation operator $\hat{O}$, the output state from beam splitter PBS$_2$ can be find as

$$|\psi_2\rangle = i\hat{O}|V\rangle$$
$$= i(-\sin \alpha|H\rangle + \cos \alpha|V\rangle)$$
$$= i \frac{1}{\sqrt{2}} \left(-\sin \alpha(|L\rangle+|R\rangle)+i \cos \alpha(|R\rangle-|L\rangle)\right)$$
$$= \frac{1}{\sqrt{2}} \left((\cos \alpha - i \sin \alpha)|L\rangle - (\cos \alpha + i \sin \alpha)|R\rangle\right).$$

Now the probability of $D_L$ clicking becomes

$$|\langle L|\psi_2\rangle|^2 = \frac{1}{2}(\cos \alpha - i \sin \alpha)(\cos \alpha + i \sin \alpha)$$
$$= \frac{1}{2} (\cos^2 \alpha - (i \sin \alpha)^2) = \frac{1}{2},$$

which is independent of $\alpha$. No interference is taking place, as one possibility has been blocked altogether.

(iv) If rotator $\hat{O}$ is removed, then

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle)$$
$$= \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}(|L\rangle+|R\rangle) - \frac{1}{\sqrt{2}}(|R\rangle-|L\rangle)\right) = \frac{1}{2} (2|L\rangle) = |L\rangle.$$

Then there is a maximum probability of $D_L$ clicking $|\langle L|\psi_2\rangle|^2 = 1$, and zero probability of $D_R$ clicking.

4. An X-ray fluorescence spectrometer analyzes the energy of an X-ray photon.

Energy $\mathcal{H}$ is an observable and the corresponding Hermitian operator is $\hat{\mathcal{H}}$. The incoming state is

$$|\psi\rangle = \sqrt{2}|1\rangle + \sqrt{3}|2\rangle + |3\rangle + |4\rangle,$$
where $|1\rangle$, $|2\rangle$, $|3\rangle$ and $|4\rangle$ are the nondegenerate eigenstates of $\hat{H}$, such that

$$\hat{H}|n\rangle = n^2\mathcal{E}_0|n\rangle,$$

and $\mathcal{E}_0$ is a constant with dimensions of energy.

(a) Normalize the input state $|\psi\rangle$. [3]

(b) Write the operator $\hat{H}$ as a sum of outer products. [3]

(c) What are the possible outcome of the energy measurement and what are their respective probabilities? [4]

(d) What is the average energy measured, when the analysis is repeated on a large number of identically prepared X-ray photons? [5]

**Answer**

(a) Since

$$\langle \psi|\psi \rangle = \left( \sqrt{2}\langle 1| + \sqrt{3}\langle 2| + \langle 3| + \langle 4| \right) \left( \sqrt{2}\langle 1| + \sqrt{3}\langle 2| + \langle 3| + \langle 4| \right)$$

$$= 2 + 3 + 1 + 1 = 7,$$

the normalized state is

$$|\psi\rangle = \frac{1}{\sqrt{7}}|1\rangle + \frac{\sqrt{3}}{\sqrt{7}}|2\rangle + \frac{1}{\sqrt{7}}|3\rangle + \frac{1}{\sqrt{7}}|4\rangle.$$

(b) Since any Hermitian operator can be spectrally decomposed, so $\hat{H}$ can be write as

$$\hat{H} = \sum_{n=1}^{4} n^2\mathcal{E}_0|n\rangle\langle n|$$

$$= \mathcal{E}_0|1\rangle\langle 1| + 4\mathcal{E}_0|2\rangle\langle 2| + 9\mathcal{E}_0|3\rangle\langle 3| + 16\mathcal{E}_0|4\rangle\langle 4|.$$
(c) Four possible outcomes of observable, the energy $\mathcal{H}$ and their respective probabilities are given below:

| $\mathcal{H}_n = n^2 \mathcal{E}_0$ | $P_n = |\langle n|\psi\rangle|^2$ |
|---------------------------------|---------------------------------|
| $\mathcal{E}_0$                | $\frac{2}{7}$                    |
| $4\mathcal{E}_0$               | $\frac{3}{7}$                    |
| $9\mathcal{E}_0$               | $\frac{1}{7}$                    |
| $16\mathcal{E}_0$              | $\frac{1}{7}$                    |

(d) The expectation value (an average) of energy $\mathcal{H}$ can be find as:

$$\langle \hat{\mathcal{H}} \rangle = \langle \psi | \hat{\mathcal{H}} | \psi \rangle.$$

Now

$$\hat{\mathcal{H}}|\psi\rangle = \left( \mathcal{E}_0 |1\rangle \langle 1| + 4\mathcal{E}_0 |2\rangle \langle 2| + 9\mathcal{E}_0 |3\rangle \langle 3| + 16\mathcal{E}_0 |4\rangle \langle 4| \right) \left( \sqrt{\frac{2}{7}} |1\rangle + \sqrt{\frac{3}{7}} |2\rangle + \sqrt{\frac{1}{7}} |3\rangle + \sqrt{\frac{1}{7}} |4\rangle \right)$$

$$= \frac{1}{\sqrt{7}} \left( \mathcal{E}_0 \sqrt{2} |1\rangle + 4\mathcal{E}_0 \sqrt{3} |2\rangle + 9\mathcal{E}_0 |3\rangle + 16\mathcal{E}_0 |4\rangle \right)$$

$$\langle \psi | \hat{\mathcal{H}} | \psi \rangle = \frac{1}{\sqrt{7}} \left( \sqrt{\frac{2}{7}} |1\rangle + \sqrt{\frac{3}{7}} |2\rangle + \sqrt{\frac{1}{7}} |3\rangle + \sqrt{\frac{1}{7}} |4\rangle \right) \left( \mathcal{E}_0 \sqrt{2} |1\rangle + 4\mathcal{E}_0 \sqrt{3} |2\rangle + 9\mathcal{E}_0 |3\rangle + 16\mathcal{E}_0 |4\rangle \right)$$

$$= \frac{\mathcal{E}_0}{7} \left( \sqrt{2} |1\rangle + \sqrt{3} |2\rangle + |3\rangle + |4\rangle \right) \left( \sqrt{2} |1\rangle + 4\sqrt{3} |2\rangle + 9 |3\rangle + 16 |4\rangle \right)$$

$$= \frac{\mathcal{E}_0}{7} \left( 2 + 4(3) + 9 + 16 \right) = \frac{39\mathcal{E}_0}{7}.$$