Assignment 1: Solution

1. Prove that if a right-circularly polarized beam of light passes through a half-wave plate, the outgoing beam becomes left-circularly polarized, independent of the orientation of the fast axis of the wave plate.

Answer

The transforming matrix (also called a Jones matrix) for a half-wave plate whose fast axis makes an angle \( \theta \) w.r.t horizontal, is given by

\[
T_{HWP} = \begin{pmatrix}
\cos 2\theta & \sin 2\theta \\
\sin 2\theta & -\cos 2\theta
\end{pmatrix}.
\]

This matrix can be derived by the knowledge that if the fast axis is horizontal, the matrix is

\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}.
\]

See its effect on the states \( |H\rangle \) and \( |V\rangle \). Then predict the effects on the states \( |\theta\rangle \) and \( |\theta + \pi/2\rangle \). We have done similar calculations in class. For a right-circularly polarized beam \( |R\rangle \) incident on a half-wave plate, the outgoing beam becomes

\[
T_{HWP}|R\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix}
\cos 2\theta & \sin 2\theta \\
\sin 2\theta & -\cos 2\theta
\end{pmatrix} \begin{pmatrix}
1 \\
-i
\end{pmatrix}
\]

\[
= \frac{1}{\sqrt{2}} \begin{pmatrix}
\cos 2\theta - i \sin 2\theta \\
\sin 2\theta + i \cos 2\theta
\end{pmatrix}
\]

\[
= \frac{1}{\sqrt{2}} \begin{pmatrix}
\cos 2\theta - i \sin 2\theta \\
i(\cos 2\theta - i \sin 2\theta)
\end{pmatrix}
\]

\[
= \frac{1}{\sqrt{2}} e^{-i2\theta} \begin{pmatrix}
e^{-i2\theta} \\
e^{-i2\theta}
\end{pmatrix} = e^{-i2\theta} |L\rangle,
\]

where \( e^{-i2\theta} \) is global phase and physically immaterial. This shows that for a right-circularly polarized beam of light passing through a half-wave plate, the outgoing beam becomes left-circularly polarized. The output is independent of the orientation.
2. (a) Compute the transforming matrix (also called a Jones matrix) for a combination of elements comprising a quarter-wave plate whose fast axis makes an angle of $+45^\circ$ from the horizontal, followed by a horizontal polarizer, followed by a quarter-wave plate whose fast axis makes an angle of $-45^\circ$ from the horizontal.

(b) If right-circularly polarized wave is incident on this combination, what will be the output polarization of the transmitted beam? Clearly show your working.

**Answer**

(a) A beam of input photons passes first through a quarter-wave plate whose fast axis makes an angle of $+45^\circ$ from the horizontal, followed by a horizontal polarizer which is followed by a quarter-wave plate whose fast axis makes an angle of $-45^\circ$ from the horizontal, as shown in Figure 1.

\[
T_{45} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}
\]

\[
T_H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

Jones matrices for a quarter-wave plate whose fast axis makes an angle of $+45^\circ$, a horizontal polarizer and a quarter-wave plate whose fast axis makes an angle of $-45^\circ$ are respectively given by

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$T_{-45} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ \hspace{1cm} (3)

and the transforming matrix for this combination of elements can be computed as

$$
T_{-45}T_H T_{+45} = \left( \frac{1}{\sqrt{2}} \right)^2 \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}
$$

$$
= \frac{1}{2} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}
$$

$$
= \frac{1}{2} \begin{pmatrix} 1 -i \\ i & 1 \end{pmatrix}.
$$

The matrices in Eqs (1),(2) and (3) can be derived or picked up from table 2.2, p.34

(b) For a right-circularly polarized incident beam on this combination, the output polarization is given by

$$
T_{-45}T_H T_{+45} |R\rangle = \frac{1}{2} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} |R\rangle
$$

$$
= \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix}
$$

$$
= \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 -1 \\ i -i \end{pmatrix} = 0,
$$

so all the photons are blocked.

3. Suppose $|\theta\rangle$ represents the state of a beam of photons linearly polarized at an angle of $\theta$ from the horizontal.

(a) Write $|\theta\rangle$ as a linear combination of $|H\rangle$ and $|V\rangle$.

(b) What is the probability that a photon in the state $|\theta\rangle$, will be measured to have vertical polarization?

(c) What is the probability that a photon in the state $|\theta\rangle$, will be measured to have linear polarization along $+45^\circ$?

(d) What is the probability that a photon in the state $|\theta\rangle$, will be measured to have right-circular polarization?
(e) If a beam of photons in state $|V\rangle$ is sent through a series of two polarization beam splitters (polarization analyzers), as illustrated in Fig. (2), then

(i) What fraction of the input photons will survive to the final output?

(ii) At what angle $\theta$ must the $PA_\theta$ be oriented so as to maximize the number of photons that are transmitted by the $PA_{HV}$? What fraction of the photons are transmitted for this particular value of $\theta$?

(iii) What fraction of the photons are transmitted if the $PA_\theta$ is simply removed from the experiment?

**Answer**

(a) A linear polarization state $|\theta\rangle$ can be expressed as a linear combination of $|H\rangle$ and $|V\rangle$:

$$|\theta\rangle = \cos \theta |H\rangle + \sin \theta |V\rangle,$$

where $\cos \theta$ and $\sin \theta$ are the probability amplitude of the basis states.

(b) To find the probability for the state $|\theta\rangle$ to have vertical polarization, first find the probability amplitude (an overlap between $|V\rangle$ and $|\theta\rangle$) expressed as the inner product

$$\langle V|\theta \rangle = \langle V| \left( \cos \theta |H\rangle + \sin \theta |V\rangle \right)$$

$$= \cos \theta \langle V|H\rangle + \sin \theta \langle V|V\rangle$$

$$= \sin \theta,$$

since $\langle V|H\rangle = 0$ and $\langle V|V\rangle = 1$.

The probability of measuring $|V\rangle$ given the state $|\theta\rangle$ is

$$P \left( V |\theta \rangle \right) = |\langle V|\theta \rangle|^2 = \sin^2 \theta.$$
(c) Using the same arguments as provided above,

$$\langle +45^\circ|\theta \rangle = \langle +45^\circ| (\cos \theta |H\rangle + \sin \theta |V\rangle \rangle.$$  

With,

$$| + 45^\circ \rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle),$$  

we obtain

$$\langle +45^\circ|\theta \rangle = \left(\frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)\right) \left( \cos \theta |H\rangle + \sin \theta |V\rangle \right)$$  

$$= \frac{1}{\sqrt{2}} \left( \cos \theta \langle H|H\rangle + \sin \theta \langle V|V\rangle \right) \cdot \langle \langle H|V\rangle = 0 = \langle V|H\rangle$$  

$$= \frac{1}{\sqrt{2}} \left( \cos \theta + \sin \theta \right),$$  

so the probability of being in $| + 45^\circ \rangle$ becomes

$$P\left( +45^\circ|\theta \right) = |\langle +45^\circ|\theta \rangle|^2$$  

$$= \frac{1}{2} \left( \cos \theta + \sin \theta \right)^2$$  

$$= \frac{1}{2} \left( \cos^2 \theta + \sin^2 \theta + 2 \cos \theta \sin \theta \right)$$  

$$= \frac{1}{2} \left( 1 + 2 \cos \theta \sin \theta \right) = \frac{1}{2} (1 + \sin 2\theta).$$  

(d) The right-circular polarization state $|R\rangle$ can be expressed as

$$|R\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle).$$  

The probability amplitude for the state $|\theta \rangle$ to have right-circular polarization can be found as

$$\langle R|\theta \rangle = \left(\frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle)\right) \left( \cos \theta |H\rangle + \sin \theta |V\rangle \right)$$  

$$= \frac{1}{\sqrt{2}} \left( \cos \theta \langle H|H\rangle + i \sin \theta \langle V|V\rangle \right) \quad \text{(other terms are zero)}$$  

$$= \frac{1}{\sqrt{2}} \left( \cos \theta + i \sin \theta \right)$$  

$$= \frac{e^{i\theta}}{\sqrt{2}}.$$  

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Therefore, the probability of being in $|R\rangle$ becomes

$$P(R|\theta\rangle) = |\langle R|\theta\rangle|^2 = \left(\frac{e^{-i\theta}}{\sqrt{2}}\right)\left(\frac{e^{i\theta}}{\sqrt{2}}\right) = \frac{1}{2}.$$ (e) The first PA$_\theta$ directs the incident beam $|V\rangle$ along two possibilities, $|\theta\rangle$ and $|\theta + 90^\circ\rangle$ and then PA$_{HV}$ directs the $|\theta\rangle$ photons into two possibilities $|V\rangle$ and $|H\rangle$.

(i) To find the fraction of input photons survive to output, we need to find the probability of $|\theta\rangle$ going through PA$_\theta$ and probability that $|\theta\rangle$ can transmit through PA$_{HV}$. First we find $|V\rangle$ as a superposition of $|\theta\rangle$ and $|\theta + 90^\circ\rangle$. It is known that

$$|\theta\rangle = \cos \theta |H\rangle + \sin \theta |V\rangle \quad (4)$$

$$|\theta + 90^\circ\rangle = \cos(\theta + 90^\circ)|H\rangle + \sin(\theta + 90^\circ)|V\rangle$$

$$= -\sin \theta |H\rangle + \cos \theta |V\rangle. \quad (5)$$

Multiply Eq(4) by $\sin \theta$ and Eq(5) by $\cos \theta$ and adding,

$$\sin \theta |\theta\rangle = \sin \theta \cos \theta |H\rangle + \sin^2 \theta |V\rangle$$

$$\cos \theta |\theta + 90^\circ\rangle = -\cos \theta \sin \theta |H\rangle + \cos^2 \theta |V\rangle,$$

so

$$|V\rangle = \sin \theta |\theta\rangle + \cos \theta |\theta + 90^\circ\rangle.$$

Proceeding ahead,

$$\langle V|\theta\rangle = (\sin \theta \langle \theta | + \cos \theta \langle \theta + 90^\circ |) |\theta\rangle$$

$$= \sin \theta.$$

Fraction of photons after PA$_\theta$ is $\sin^2 \theta$. To find the fraction of $|H\rangle$ photons after PA$_{HV}$, we calculate,

$$\langle H|\theta\rangle = \langle H| (\cos \theta |H\rangle + \sin \theta |V\rangle)$$

$$= \cos \theta,$$

and $|\langle H|\theta\rangle|^2 = \cos^2 \theta$. Hence the fraction being transmitted is $\sin^2 \theta \cos^2 \theta$.

(ii) The number of photons $N_{\text{photons}}$ will be maximum when $\sin^2 \theta \cos^2 \theta$ is maximum. Now

$$\sin^2 \theta \cos^2 \theta = \frac{1}{4} (\sin 2\theta)^2,$$
where \( \sin 2\theta \) is maximum for \( \theta = \pi/4 \). Therefore, \( \text{PA}_\theta \) must be at \( (\pi/4) \) to maximize the number of photons. This fraction is,

\[
N_{\text{photons}} = (\sin 45 \cos 45)^2 = \frac{1}{4}.
\]

(iii) If \( \text{PA}_\theta \) is simply removed as shown in Figure 3,

then all photons are blocked because the probability of beam \( |V\rangle \) to pass through \( \text{PA}_{HV} \) vanishes, \( |\langle H|V\rangle|^2 = 0 \) since \( \langle V|H\rangle = 0 \).

4. (a) A stream of photons entering the interferometer of Fig. (4) is in the state \( |V\rangle \). Would you expect to see interference? Write the transforming matrices and quantum states at each step of the experiment.

(b) If the input photons are in the left circularly polarized state \( |L\rangle \), do you expect to see quantum interference? Show your working.

**Answer**

(a) The transforming matrices can be computed as,

\[
\text{PA}_{(HV)}_1 = \begin{pmatrix}
e^{i\gamma} & 0 \\
0 & 1
\end{pmatrix} = T_A
\]

\[
\text{QWP with fast axis at } 45^\circ = \frac{1}{\sqrt{2}} \begin{pmatrix}1 & -i \\
-i & 1\end{pmatrix} = T_B
\]

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Now the quantum states at each step of this experiment can be found as follows.

After first beam splitter:

\[ T_A|V\rangle = \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |V\rangle \]

After quarter-wave plate:

\[ T_B \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} \]

After second beam splitter:

\[ T_A \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} = \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -i \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -ie^{i\gamma} \\ 1 \end{pmatrix} \]

\[ = \frac{1}{\sqrt{2}} (-ie^{i\gamma}|H\rangle + |V\rangle) = |\psi_I\rangle. \]

The last PA\textsubscript{45} prepares two possibilities |45\degree\rangle and |−45\degree\rangle. We can find the probability amplitudes:

\[ \langle 45\degree|\psi_I\rangle = \frac{1}{\sqrt{2}} ((|H\rangle + |V\rangle) \cdot \frac{1}{\sqrt{2}} (-ie^{i\gamma}|H\rangle + |V\rangle) \]

\[ = \frac{1}{2} (-ie^{i\gamma} + 1) \]

\[ |\langle 45\degree|\psi_I\rangle|^2 = \frac{1}{2} (ie^{-i\gamma} + 1) \frac{1}{2} (-ie^{i\gamma} + 1) \]

\[ = \frac{1}{4} (1 + ie^{-i\gamma} - ie^{i\gamma} + 1) \]

\[ = \frac{1}{4} (2 - i(2i\sin \gamma)) = \frac{1}{2} (1 + \sin \gamma) = \sin^2 \left( \frac{\gamma}{2} \right). \]

Similarly one can show that |⟨−45\degree|\psi_I⟩|^2 = 1 − sin^2(\frac{\gamma}{2}) = cos^2(\frac{\gamma}{2}). The relative phase difference between |H\rangle and |V\rangle in |\psi_I\rangle results in a phase dependent output.

(b) Using the same procedure of part (a), for the incident photons in left circularly polarized state |L\rangle:

After first beam splitter:

\[ T_A|L\rangle = \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\gamma} \\ i \end{pmatrix} \]

After quarter-wave plate:

\[ T_B \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\gamma} \\ i \end{pmatrix} = \left( \frac{1}{\sqrt{2}} \right)^2 \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} e^{i\gamma} \\ i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{i\gamma} + 1 \\ -ie^{i\gamma} + i \end{pmatrix} \]
After second beam splitter:

\[
T_{A_T/2} \frac{1}{2} \begin{pmatrix}
eg i^2 + 1 \\
eg i e^{i\gamma} + i 
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
eg i^2 + 1 \\
eg i e^{i\gamma} + i 
\end{pmatrix} \frac{1}{2} \begin{pmatrix}
eg i^2 + 1 \\
eg i e^{i\gamma} + i 
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
eg i^2 + 1 \\
eg i e^{i\gamma} + i 
\end{pmatrix} = \frac{1}{2} \left( (e^{i\gamma} + e^{i\gamma}) |H\rangle + (-ie^{i\gamma} + i)|V\rangle \right) = |\psi_t\rangle.
\]

The last PA_{45} prepares two possibilities |45°\rangle and |−45°\rangle. Using the same method as we did in part in (a), the probability amplitudes can be find as:

\[
\langle 45^\circ |\psi_t \rangle = \frac{1}{\sqrt{2}} (\langle H | + \langle V |) \frac{1}{2} \left( (e^{i\gamma} + e^{i\gamma}) |H\rangle + (-ie^{i\gamma} + i)|V\rangle \right)
\]

\[
= \frac{1}{2\sqrt{2}} \left( e^{i\gamma} + e^{i\gamma} - ie^{i\gamma} + i \right)
\]

\[
|\langle 45^\circ |\psi_t \rangle|^2 = \left( \frac{1}{2\sqrt{2}} \right)^2 \left( e^{-2i\gamma} + e^{-i\gamma} + ie^{-i\gamma} - i \right) \left( e^{2i\gamma} + e^{i\gamma} - ie^{i\gamma} + i \right)
\]

\[
= \frac{1}{8} \left( 4 - i(2i) \left( \frac{e^{2i\gamma} - e^{-2i\gamma}}{2i} \right) \right)
\]

\[
= \frac{1}{4} (2 + \sin(2\gamma)).
\]

Similarly, one can show that |\langle -45^\circ |\psi_t \rangle|^2 = \frac{1}{4} (2 - \sin(2\gamma)).