The Magnetic Torque Oscillator and the Magnetic Piston

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A magnet suspended in a uniform magnetic field like that of the Earth can be made to oscillate about the field. The frequency of oscillation depends on the strength (magnetic moment) of the magnet, that of the external field, and the moment of inertia of the magnet. It is easily shown and verified by experiment that a simple but nontrivial expression represents this motion well, but that only the product of the magnetic moment and magnetic field can be determined. The repulsion of two magnets can be used to determine the magnetic moment, and in turn the external field can be determined. The results are reasonably good considering the very simple equipment required, and the experiment allows quantitative investigation of magnetism.

A magnet's strength is well-indicated by its magnetic moment μ. When placed in an external magnetic field $\mathbf{B}$, a torque τ arises on the magnet[1] to align it with the external field. This effect is well-known since it is what makes a compass needle line up with magnetic north. More precisely, the torque is equal to $\tau = -\mu B \sin\theta$, where θ is the angle between the external field and the magnetic moment vector $\boldsymbol{\mu}$ (directed from the south pole to north pole in a regular magnet). From the angular form of Newton's second law $\tau = I\alpha = I(d^2\theta/dt^2)$, and assuming small angles ($\theta \approx 10^{\circ}$ or less), we can write the equation of motion for the magnet as

$$\frac{d^2\theta}{dt^2} = -\left(\frac{\mu B}{I}\right)\theta,$$

where I is its moment of inertia about the suspension axis. This is the equation for a simple harmonic oscillator with angular frequency[2] $\omega = \sqrt{\mu B/I}$. So, much like a simple pendulum in a gravitational field, a magnet will oscillate in the magnetic field if properly suspended. This oscillation may be easily seen in a compass, but in that application it is not desired. Therefore, good compasses are liquid-filled to damp out any movement and leave the needle aligned with the horizontal component of the Earth's magnetic field.

Fig. 1. Two magnets attracting each other with a thread between them do not hang vertically due to their attempt to align with the Earth's magnetic field, which is not, in general, horizontal. The level is made of aluminum (a protective cover over the bubble was removed as the screws holding it in place were magnetic).

A useful, if often hidden, aspect of modern technology is that strong magnets may be produced cheaply. Most earphones, and all hard drives, use

DOI: 10.1119/1.2783155

strong magnets, so it is very difficult not to be surrounded by strong magnets in modern life. In their "raw" form, small strong magnets may be purchased in hardware stores.[3] If a pair of such strong cylindrical magnets is carefully brought together with a thread placed between them, as shown in Fig. 1, the thread will be held in place between them, and the magnets may be suspended in space and react to Earth's magnetic field. The torque referred to above will make them north-seeking, much like a compass needle. Since there is very little damping, oscillation (usually present or easily started) about the (nearly) vertical axis formed by the thread is quite striking and will continue for a very long time. A more subtle aspect is that the magnet attempts to align with Earth's field in three dimensions. Alignment parallel to the vertical part of the field, present everywhere except the magnetic equator, is not possible due to the weight of the magnet.[4] Nevertheless, as Fig. 1 illustrates, the magnets do not hang vertically. Since it is the intent of this experiment to determine the horizontal component of Earth's magnetic field, the thread must be weighted to hang more vertically. It is also important that the thread not have any twist in it that would produce a torsional torque. In this experiment, a tape measure weighing about 500 g was tied to the bottom of the thread about one meter below the magnets and the thread suspended from a tack near ceiling level.

The small magnets used here are disks (i.e., short cylinders) that rotate about a central diameter. The appropriate formula for the moment of inertia, which depends on the mass M, radius R, and height H of the cylinder, can be found in some introductory physics texts[5] as $I = \frac{1}{12}MH^2 + \frac{1}{4}MR^2$. In our case, we vary I by adding more magnets of the same radius and thickness, so the moment of inertia should increase with the square of the number of magnets stacked.

We must make an assumption in order to determine how the magnetic moment μ varies as magnets are stacked. When a magnetic field is applied to many magnetically susceptible materials, randomly oriented "domains" in the material align and increase the magnetic moment.[6] A ferromagnetic material as found in a strong magnet may be regarded as an extreme case of a paramagnetic material in that it is saturated; we assume that further application of a magnetic field will not increase the alignment of domains as they are already very highly aligned. Thus we can assume that the overall magnetic moment simply increases directly as the number of magnets.

Using N magnets, each of thickness h, mass m, and magnetic moment μ_1, the formula for the oscillation frequency becomes

$$\omega(N) = \sqrt{\frac{\mu_1 B}{m}} \sqrt{\frac{1}{\frac{1}{12}h^2N^2 + \frac{1}{4}R^2}}. \tag{1}$$

This formula allows ready testing. By adding magnets to each side of the string, N can be varied in units of 2. In our case, N was varied from 2 to 14. Since larger sources of error were expected, the other required quantities of magnet mass and dimensions were determined somewhat roughly with a postal scale and ruler. Twelve magnets had a mass of 80 g, thus m was 0.0666 kg. Their length when together was 3.8 cm, so h was 0.00317 m. The diameter was 18.5 mm by direct measurement, so radius R was 0.00925 m.

Multiple oscillations of the stacked magnets were observed and timed using a wristwatch with a stopwatch setting. When two magnets were used, 100 oscillations were timed and 74 s elapsed; with 14 magnets, 20 oscillations took slightly over 40 s. The period P is simply the time elapsed divided by the number of oscillations. Using Vernier's Graphical Analysis[7] the period P data values were entered, the frequency was determined as a new calculated column ($2\pi/P$), and a curve fit was done based on Eq. (1). The results are shown in Fig. 2, where the somewhat complex curve passes very near all data points.

The good fit confirms that the basic theory behind Eq. (1) is correct: magnetic moments add when magnets stick together, and the moment of inertia behaves as it should. The only parameter in the fit is referred to as A in the fit function, and its value of 0.00185 corresponds to the product $\mu_1 B$ in SI units.[8] In order to disentangle these values, one of them must be determined. A simple way to do this is to balance magnetic force against gravitational force. For two magnets approximated as interacting dipoles (magnetic moments), both forces may be easily calculated, and this is shown below.

First, since this paper tries to show how to do things simply and inexpensively, we show how to re-

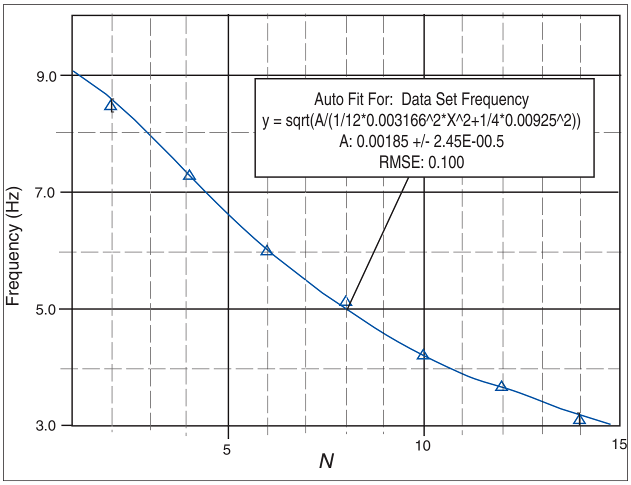

Fig. 2. Graphical Analysis fit. Data points show the frequency of oscillation as a function of the number of magnets. The fit corresponds to Eq. (1) and has only one parameter. The good fit using a complex curve indicates that the theory is basically correct.

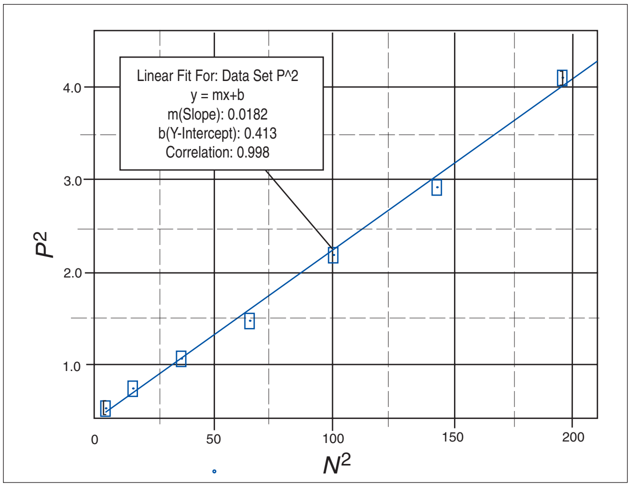

Fig. 3. Data plotted in the linearized form corresponding to Eq. (3) in the text. The representation by a straight line is very good and analysis could be done using graph paper.

vise Eq. (1) so that a straight line can be plotted to allow parameters to be determined. The dependence on N is made more explicit if it is taken to the numerator by using the period P as the independent variable:

$$P(N) = 2\pi\sqrt{\frac{m}{\mu_1 B}}\sqrt{\tfrac{1}{12}h^2N^2 + \tfrac{1}{4}R^2}. \qquad (2)$$

This equation can be made linear by squaring both sides to get

$$P^2(N^2) = \frac{\pi^2 m}{\mu_1 B}\left(\tfrac{1}{3}h^2N^2 + R^2\right). \qquad (3)$$

We have plotted our data in this form of P^2 as a function of N^2 (for convenience using Graphical Analyis). The results and a straight-line fit are created in Fig. 3. A correlation coefficient of 1 indicates a straight line, so the correlation coefficient of 0.998 determined in linear fitting with Graphical Analysis supports the linear relation shown by Eq. (3). In turn, as already shown with the nonlinear form, this implies that the theory is very accurate and analysis should be reliable. For simplicity we will base further analysis on the parameter already derived using Eq. (1). Those who do not have Graphical Analysis should be able to readily figure out how to do a linear fit on graph paper and from it derive the needed values using Eq. (3).

Magnetic Dipole Moment from Magnetic Piston

By making a "magnetic piston," the magnetic moment can be found with very little instrumentation. We saw above that a uniform magnetic field exerts a torque on a dipole (and in fact it exerts no force; a compass needle does not try to pull the user toward the magnetic pole, only to orient itself). Obviously, magnets also exert forces, and this is characteristic of nonuniform magnetic fields. Small magnets have complex (and nonuniform) fields close to them. If the magnets are far enough apart, their interaction can be regarded as that of dipoles. Modern strong magnets allow big enough spacing that the dipole approximation is good while significant force is present.[9] This allows us to arrange magnets carefully in order to balance magnetic force against gravity. In this way, with an expression for the force between dipoles, we can determine the dipole moment. A narrow clear plastic tube allows placing one stack of magnets (one magnet alone would easily rotate) above another without rotation from magnetic torque taking place. In this way, the upper magnet stack will float in the air above the lower one. This setup can be referred to as a "magnetic piston" and is illustrated in Fig. 4.

Along the axis of a dipole at the origin pointing along the z direction, the field is given[10] by

$$B_z = \frac{\mu_0}{4\pi}\frac{2\mu}{z^3}, \tag{4}$$

where $\mu_0 = 4\pi\times10^{-7}[T\cdot m/A]$. This field is nonuniform even along the axis since it falls off rapidly. The force is simply equal to the gradient of the magnetic field times the component of the dipole moment μ parallel to z and is along that axis. For two identical magnets of moment μ, we find that the magnitude of the force is

$$F_z = 3\frac{\mu_0}{4\pi}\frac{2\mu^2}{z^4}.$$

The magnetic piston was made by taping four stacked magnets to a tabletop and placing another four inside a tight plastic tube where they could freely slide up and down, with the tube taped in place above the other four magnets so that the magnets in the tube were suspended by magnetic force. In this configuration, 5.7 cm separated the faces of the floating magnets from those below. The z value appropriate for the centers of the magnets, which are the effective dipole positions, is 6.3 cm. The repulsive magnetic force upward must have been equal to the gravitational force on the floating magnets. Thus

$$4mg = 3\frac{\mu_0}{4\pi}\frac{2(4\mu_1)^2}{z^4},$$

and we can solve for the magnetic moment of a magnet as

$$\mu_1 = \sqrt{\frac{10^7 mg}{3}}z^2 = 0.661 A\cdot m^2.$$

This is comparable to, but slightly smaller than, the value found for a similar magnet using a coil.[3] The fit in the previous section gave a value for the product $\frac{\mu_1 B}{m}$ of 0.00185, so $B = 1.9 \times 10^4$ nT.[12] Field calculators online[13] show an expected horizontal component of 14,480 nT, so our value is likely about 30% too large. Some of the error in this approach comes from the sensitivity of the force equation for interacting dipoles so that small errors in measuring and approximation of positions of magnets, as those of dipoles, result in errors in z in an equation that is quite sensitive to z. Assuming this is the only error and is roughly 3 mm, the error equation

$$\frac{\Delta\mu_1}{\mu_1} = \frac{2\Delta z}{z}$$

Fig. 4. The "magnetic piston" was made using two magnet stacks and a clear plastic coin tube. The lower stack of four magnets is taped to the tabletop to prevent rotation due to magnetic torque from the floating upper stack. The walls of the plastic tube (also taped securely) prevent significant rotation of the floating stack. A finger may be inserted to push down on the upper stack and force it to move until it comes to a force balance position.

would lead us to expect only a 10% error. It is thus likely that there was some influence of the vertical component of the Earth's field, which at the latitude this experiment was performed is much larger than the horizontal component. If this is the case, the experiment should work better at lower latitudes than those of central Canada, where our run was done, since the Earth's magnetic field would be more horizontal. It would also be possible to devise a better form of mounting if a machine shop is available, perhaps taking inspiration from historical instruments.[4] The intent of this paper was to show how to get meaningful results with the simplest possible equipment, so the inaccuracy is not of much concern here.

Conclusions

Using only very simple equipment, one can show that the form of the equation for a magnetic oscillator is correct. The period of such an oscillator determines only the product μB, but if another method can be found to determine the magnetic dipole moment, then the horizontal component of Earth's magnetic field can be determined. With a simple "magnetic piston," the dipole moment may be determined but is subject to nonnegligible error. Using it to subsequently determine Earth's field resulted in a value comparable to that expected. This experiment allows measurement techniques and otherwise nonintuitive physical quantities to be investigated at very low cost.

References

1. The torque on a coil is derived in Chap. 20 of D. Giancoli, *Physics*, 6th ed. (Pearson Prentice Hall, Upper Saddle River, NJ, 2005), and the relation to permanent magnets is explained.
2. This result follows immediately by solving the equation of motion with calculus; a noncalculus discussion of various types of harmonic oscillators is found in Chap. 13 of J. Touger, *Introductory Physics: Building Understanding* (Wiley, New York, 2006).
3. See the recent *TPT* article by Mike Moloney, "Strong little magnets," *Phys. Teach.* **45**, 352–355 (Sept. 2007) and an earlier, related article by G. Hageseth, *Am. J. Phys.* **37**, 529–531 (May 1969). The magnets used in this experiment were purchased at Lee Valley Tools' local store, but similar magnets are now widely available.
4. If allowed to pivot freely about the center of mass, a dip indicator would result. These were historically used as scientific instruments, including on expeditions to find the north magnetic pole (the "dip pole"). Usually the form was like a vertical compass or "dip needle". See http://physics.kenyon.edu/EarlyApparatus/Electricity/Dip_Needle/Dip_Needle.html .
5. See, for example, D. Halliday, R. Resnick, and J. Walker, *Fundamentals of Physics*, 8th ed. (Wiley, New York, 2007), p. 253.
6. This is known as paramagnetism; the opposite behavior, diamagnetism, is also possible. Deeper understanding of magnetism is based on advanced concepts. Reading accessible to those slightly beyond the first year level can be found in the *Feynman Lectures on Physics*, Vol. II (Addison Wesley, Reading, MA, 1964).
7. http://www.vernier.com gives information about this inexpensive and useful program.
8. If SI units are consistently used, the details of the complex magnetic units can be avoided.
9. The magnetic field very near the magnet is in general quite complex, although it may have some simplifying characteristics such as symmetry
10. This is a modification of the spherical coordinate form given in P. Lorrain and D. Corson, *Electromagnetic Fields and Waves*, 2nd ed. (Freeman, New York, 1970) in Chap. 7. A more exact form based on the magnetization of the magnet and its dimensions is given by M. Connors, "Measurement and analysis of the field of disk magnets," *Phys. Teach.* **40**, 308–311 (May 2002) and could be reworked to allow μ_1 to be determined by measuring the magnetic field along the axis. However, the intent here is to use minimal instrumentation.
11. J.D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975), p. 185.
12. Earth field measurements are usually given in units of nanotesla (nT).
13. http://gsc.nrcan.gc.ca/geomag/field/magref_e.php leads to the International Geomagnetic Reference Field and also the version specific to Canada.

PACS codes: 01.50.Pa, 07.00.00

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