Lab Monograph
Introductory Experimental Physics

Muhammad Sabieh Anwar
Sohaib Shamim
Wasif Zia
and
Waqas Mahmood
Contents

1 Working in the lab and reading the lab manuals 1

1.1 Learning outcomes ........................................... 1
1.2 Website ......................................................... 2
1.3 Mathematical computing ..................................... 3
1.4 Error analysis .................................................. 3
1.5 Data acquisition ............................................... 3
1.6 Our experiments ............................................... 4
1.7 Guidelines for practical work inside the Lab .......... 5
1.8 How to read the Lab manual ............................... 6
1.9 Grading and assessment ...................................... 9

2 Error Analysis in the Experimental Physics Lab 11

2.1 How to report and use uncertainties .................... 11
2.2 Types of errors ............................................... 12
2.3 Quantifying errors in measurements ................... 15
2.4 Checking Relationships with a Graph ................ 17
2.5 Propagation of Uncertainties ......................... 18

3 Questions for Error Analysis 22
4 Solution key for error analysis problems 25

5 Introduction to Matlab for Experimental Physics 32
   5.1 Vectors and Matrices ........................................ 33
   5.2 Graphs and Plotting ........................................ 41
   5.3 Curve Fitting ............................................... 51

6 Simple Harmonic Motion is Observed through Webcam 64
   6.1 Conceptual Objectives ..................................... 64
   6.2 Experimental Objectives .................................. 65
   6.3 Theoretical Introduction ................................. 65
   6.4 Apparatus ............................................... 70
   6.5 The Experiment .......................................... 70
   6.6 Experience Questions ................................. 74
   6.7 Idea Experiments .................................... 74

7 Rotational Dynamics, Moment of Inertia, Torque and Rotational Friction 76
   7.1 Conceptual Objectives .................................. 77
   7.2 Experimental Objectives .............................. 77
   7.3 Theoretical Introduction ................................. 77
   7.4 Introduction to the Apparatus ..................... 82
   7.5 Experimental Method ..................................... 84
   7.6 Experience Questions .................................. 89
   7.7 Idea Experiments .................................... 89

8 Heat Transfer and Newtons Law of Cooling 90
   8.1 Conceptual Objectives .................................. 91
CONTENTS

8.2 Experimental Objectives ............................................ 91
8.3 Theoretical Introduction ........................................... 91
8.4 Apparatus ............................................................. 97
8.5 Experimental Method ............................................... 99
8.6 Experience Questions .............................................. 103
8.7 Idea Experiments .................................................... 103

9 Magnetic Phase Transitions of a Ferromagnetic Alloy ............. 105

9.1 Conceptual Objectives .............................................. 106
9.2 Experimental Objectives ........................................... 107
9.3 Theoretical Introduction .......................................... 107
9.4 Apparatus and Experimental Preparation ......................... 112
9.5 Experimental Method .............................................. 116
9.6 Experience Questions .............................................. 119
9.7 Idea Experiments .................................................... 119

10 Optical Activity of the Chiral Solutions .......................... 121

10.1 Conceptual Objectives ............................................. 122
10.2 Experimental Objectives ......................................... 122
10.3 Theoretical Introduction ......................................... 122
10.4 Apparatus .......................................................... 129
10.5 Experimental Method .............................................. 131
10.6 Experience Questions .............................................. 134
10.7 Idea Experiments .................................................... 134

11 Data Acquisition and Filter Design ............................... 136

11.1 Conceptual Objectives ............................................. 136
11.2 Experimental Objectives ......................................... 137
11.3 Introduction to the History of Electronics ...................... 137
11.4 Breadboard Layout and its Internal Connections .............. 139
11.5 Data Acquisition System ......................................... 140
11.6 Understanding the Frequency Concept ........................ 142
11.7 Verifying the Nyquist Theorem ................................... 143
11.8 Logic Gates Exemplified by the XOR Gate ..................... 147
11.9 Filter Design ...................................................... 150
11.10 Filtering a Composite Signal .................................... 156
11.11 Idea Experiments ................................................ 159

12 Latent Heat of Vaporization of Liquid Nitrogen and Specific Heats of Metals 161

12.1 Conceptual Objectives ............................................. 161
12.2 Experimental Objectives .......................................... 162
12.3 Equipartition of energy ............................................ 162
12.4 Experimental preparation and safety measures ............... 167
12.5 The Experiment ................................................... 169

13 Electromagnetic Induction and Read-Write Operations in Magnetic Media 174

13.1 Conceptual Objectives ............................................. 174
13.2 Experimental Objectives .......................................... 175
13.3 The Magnetic Field $\mathbf{B}$ and Flux $\Phi$ ........................ 175
13.4 Electromagnetic Induction ........................................ 176
13.5 Solenoids .......................................................... 177
13.6 The Hall effect ..................................................... 178
13.7 Data Storage on a hard disk ......................... 180
13.8 The Experiment .................................. 181
13.9 Hard disk operation .............................. 184

14 Vibrations on a String and Resonance .................. 190

14.1 Conceptual Objectives ............................... 190
14.2 Experimental Objectives ............................ 191
14.3 Waves and their different types .................. 191
14.4 Wave Interference and Resonance ................ 193
14.5 Wave Speed ....................................... 198
14.6 Experimental setup ............................... 200

15 Natural Radioactivity and Statistics .................. 206

15.1 APPROXIMATE PERFORMANCE TIME 1 WEEK ........ 207
15.2 Conceptual Objectives ............................... 207
15.3 Experimental Objectives ............................ 207
15.4 Theoretical Introduction ......................... 208
15.5 The Apparatus .................................... 210
15.6 Experimental Method .............................. 211
15.7 Experience Questions .............................. 214
15.8 Idea Experiments ................................. 215
Acknowledgements

_in the name of Allah, the most Beneficent and Merciful._

We would like to thank our colleagues who contributed in different ways to these experiments. Their names, along with the experiments or write-ups to which they made significant contribution are given here.

_Umer Hassan_ for experiments on data acquisition and filter design and magnetic fields, electromagnetic induction and magnetic media. _Muhammad Wasif_ for the experiment on electromagnetic induction and magnetic media. _Hafsa Hassan_ for the experiment on natural radioactivity and statistics.

We also appreciate the indefatigable help of Rabiya Salman, Junaid Alam, Asma Khalid, Amrozia Shaheen, Muhammad Wasif, Umer Hassan, Rameez Ahmad, Imran Hanif and Ahmed Waqas Zubairy for the management, commissioning, improvement and delivering the Experimental Physics 1 course for three consecutive years to approximately 220 students. These students belonged to the LUMS School of Science and Engineering. Ali and Yousaf are our Field Assistants who incessantly support us in technical and managerial tasks. Our technician Abdul Mannan contributed to the building of some of the setups. Some of the manufacturing was performed by machinists engaged from Lahore’s rich treasure of skilled craftsmanship. Finally, we thank all the students who have performed these experiments or are currently engaged in learning physics through experiments.
Chapter 1

Working in the lab and reading the lab manuals

Sabieh Anwar

1.1 Learning outcomes

Experimental Physics 1 is a modern, lab-based physics course where all the SSE students will be exposed to a variety of techniques, concepts and skills in the experimental sciences. At the end of the course, the students will be able to do the following.

*Mathematical and physical*

1. Demonstrate a keen appreciation of physical quantities, their dimensions and units.

2. Perform simple statistical analysis of data including calculating means, mean squares and correlations.

3. Mathematically understand physical processes and corroborating them with linear, exponential, sinusoidal and polynomial models.

4. Accurately represent experimental data in the form of tables and graphs.

5. Understand errors, uncertainties and their propagation.

6. Demonstrate the ability to present an idea in the following equivalent forms:
2.CHAPTER 1. WORKING IN THE LAB AND READING THE LAB MANUALS

- equations and formulas
- words
- graphs
- pictures

7. Develop an appreciation of energy, its myriad manifestations and interconversion.

Engineer ing and practical

1. Design simple experiments to test physical ideas.

2. Understand the significance of various kinds of materials (ceramics, plastics, metals, conductors, insulators) in the design of hardware.

3. Perform experiments safely.

4. Demonstrate the ability to work in teams.

5. Use locally available resources including materials and craftsmanship to build new projects.

1.2 Website

The course website is

http://physlab.lums.edu.pk

You are responsible for visiting the website regularly, at least twice a week. Announcements and course information will be uploaded here. The website contains a list of experiments as well as the following supporting material.

1. Lab manuals

2. Matlab codes

3. Labview VI’s (virtual instruments)

4. Further reading and references

5. Representative results and graphs

6. Supporting literature for the hardware
The website also displays the course time table, the lab allocation schedule, a brief history and philosophy and contact details for the lab staff.

<table>
<thead>
<tr>
<th>Week 2</th>
<th>Error analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Week 3-4</td>
<td>Mathematical computing</td>
</tr>
<tr>
<td>Weeks 6-15</td>
<td>Experiments</td>
</tr>
</tbody>
</table>

### 1.3 Mathematical computing

The laboratory course uses MATLAB as the computing environment. All students are expected to gain a minimum level of proficiency in MATLAB. For this purpose, we have dedicated the second week to a lecture based demonstration of MATLAB followed by practice labs. The practice labs will be conducted by the lab instructors on 07-10 September, 14-17 September. The schedule is posted on the website.

### 1.4 Error analysis

Errors and uncertainties lie at the heart of experimental physics. The second week is devoted to a lecture based exercise on predicting and analyzing errors. Refer to the online time table for more details.

### 1.5 Data acquisition

Some of our experiments use data acquisition (DAQ) hardware and software. On the overall, you will be expected to treat the DAQ systems as black boxes, although the more energetic and forward-looking students will be attracted towards a deeper understanding of DAQ. We are here to help you when and as you desire. Furthermore, the experiment on Electronics, in fact, also serves as a veiled introduction to DAQ.
1.6 Our experiments

For the pioneering LUMS SSE class, we have developed a set of eight experiments. These cover the following subject areas.

1. General Physics
2. Mechanics
3. Heat and Thermodynamics
4. Optics
5. Electricity and Magnetism
6. Electronics
7. Cryogens and Cryostats
8. Electromagnetic Induction and Magnetic Media

Figure 1.1: Typical student run-through the experiments. The ordering will vary from student to student. For your personal instance of allocation, visit the lab website.

You are required to perform all eight experiments, one experiment per week. Besides the general physics experiment, you will work in pairs. The lab staff has already published an allocation schedule on the website. You are not allowed to change the groupings and allocations. A typical navigational route for performing the eight experiments is shown in Figure 1.1.
1.7 Guidelines for practical work inside the Lab

The Experimental Physics 1 Lab is an intensive, high-enrollment and busy experimental environment. So I have come up with a number of guidelines that are aimed at enhancing the overall value of this course. Here are some guidelines you must all follow.

1. *Lab manual:* You must actively read the lab manual before coming for the experiment (See the next Section). You are also required to bring the manual with you to the lab.

2. *Logging into the PC's:* The lab is equipped with personal computers fitted with DAQ hardware and MATLAB. Login with your individual username and password but if you are working in pairs, then you must login with the group username and password. These details are provided on the website and also posted on the Lab noticeboard.

3. *Lab notebooks:* Each student *must* bring his/her notebook to the lab. This is a hard-bound notebook that will serve as a valuable reference in your future years. Note down the answers to all the queries (see next Section) in your notebook. The demonstrators will mark and sign your solutions only if they are presented in the proper notebook. Loose paper will not be entertained. Sample notebooks are available with the laboratory staff.

4. *Printing:* Printers are available in the lab. You must plot your graphs and paste them into your notebooks. These printers must not be used for printing lab manuals, which are available online or can be purchased from Gestetner Photocopier Centre inside LUMS.

5. *Dialogue with the demonstrators:* The experiments will be supervised by a team of lab demonstrators and instructors. The lab manuals are written to encourage dialogue. It is mandatory that you engage with the demonstrators in a meaningful two-way rapport. Never be shy of talking to your colleague or to the demonstrator. Intelligent conversations with the dialogists will help them assess your contribution to the experiment and furthermore, will contribute towards your overall grade.

6. *Role of the demonstrator:* The role of the demonstrator in the lab can be summarized in pointwise fashion.

   • Asking questions and injecting the experimenters with mental conflicts that will, hopefully, guide them towards a better understanding of the
experiment.

- Engaging the experimenters in a meaningful dialogue about the experiment.
- Attesting and marking the experimental results as and when they become available.
- Ensuring that experimenters follow all safety protocols.
- Organizing mini-tutorials before the experiment for the initiated students.
- Describing key features of the apparatus.
- In general, holding everything together.

7. **The success and failure of experiments**: It is quite likely that during the course of your lab work, some experiment might not work. Don't conceive this as a failure of the experiment. In real life, experiments seldom work in the first attempt. So as long as you can document what went wrong, and interpret the results, you will be fine. In fact, recognizing and interpreting procedural mistakes or limitations in the hardware makes you deserving of extra credit.

8. **Lab safety**: Lab safety is of paramount importance in this course. Our experiments present five kinds of hazards.

- Intense light sources called lasers.
- Hot surfaces approaching 400°C.
- Lifting and transporting hot and heavy objects.
- Large electric currents.
- Low temperatures –200°C.

Sufficient engineering controls are in place that will protect you against these hazards. Personal protective equipment (laser goggles, thermal gloves, insulation gloves and footwear) are also provided as the next level of safety. It is unacceptable not to follow the rules and in case of intentional carelessness, I reserve the right to bar your admission to the lab, to say the very least.

### 1.8 How to read the Lab manual

Each experiment comes with a Lab manual. As a pre-lab exercise, the student must read the entire lab manual (for the forthcoming experiment). These man-
uals are available from the website. They can either be downloaded and printed (on personal printers or printers located in the IST) or the consolidated booklet containing all lab manuals can be purchased from the Gestetner Photocopyers inside LUMS.

The manuals must be read **actively**, diligently and carefully. I must digress to explain what this means. You must have come across the phrase “to read between the lines”. As you read our lab manuals, you must also develop the habit of “writing between the lines”. I assure you that this is not an act of mutilation, but of love. Mark your manuals, take notes, repeat the calculations and practice all derivations. Ask yourself questions. Find answers. Take visuo-psychological leaps of imagination and make pictures, many, many pictures. Even try sketching the graphs. In this way, you may end up predicting the outcome even before the experiment.¹

You are also required to bring a copy of the manual to the lab.

The lab manuals are divided into the following sections.

1. **General introduction**: This is a prelude to the experiment. It gets you started.

2. **Keywords**: This is a listing of the main concepts and hardware used in the experiment. The purpose of this vocabulary is to provide you with a distilled extract of terms that you can quickly look up in a standard textbook, a reference book or on Google.

3. **Approximate performance time**: This is self-explanatory. Our experiments are long, but it is imperative that you do not rush through them. But do not proceed at zero speed.

4. **Conceptual objectives**: The learning outcomes of the experiment are described in the “conceptual objectives” section. These are the “take-home lessons” from the experiment. A few years down the line, you may forget the precise details of the practical exercise, but I expect you to fully assimilate the conceptual objectives. These skills and concepts will become a part of your academic personality, running in your bloodstream.

5. **Experimental objectives**: This is a concise statement of the goal of the experiment, such as “this experiment determines the optical activity of

¹If you want to know more about active, thoughtful and two-way reading habits, consult the book “How to read a book?” (1940) by Mortimer Adler.
6. *Theoretical introduction:* The section titled “theoretical introduction” summarizes the background theory required for a complete understanding of the experiment. Whenever you are in a state of confusion or yearning to know for more, consult either the References at the end of the lab manual or your favourite textbook such as *Physics for Engineers and Scientists* (H. C. Ohanian).

There are two kinds of activities, that we call *queries*, imbedded in the manuals. These queries are colour coded. inquiry, question or problem that you must solve at home, before coming to the lab. You will not be assessed on these queries, but your complete understanding of the experiment *does* depend on successfully tackling them. for a query that is activity-based, and can only be addressed inside the lab. These queries are based on the experiment, the apparatus or the data acquired therefrom. Your overall assessment and grading will depend on how you approach these queries. Since there will be black-and-white printing and photocopying of our lab manuals, I have also identified these queries by placing a ★.

7. *Apparatus:* As the name implies, this section describes the hardware used in the experiment. I have tried to put in photographs where possible. The aim is to help you *visualize* and draw mental pictures of the experiment. The sources of the equipment are also mentioned and helps give a flavour of what it takes to build teaching equipment.

8. *Experimental method:* This section forms the crux of the manual. It takes you through the experiment, step by step and query by query. On the day of the experiment, you will spend almost all of your time forging through this part of the manual.

9. *Experience questions:* This is a list of nudge questions. Ponder over these questions in your free time. The physics lab staff will be happy to discuss these points with you.

10. *Idea experiments:* The poet William Blake once remarked,

    I will not reason and compare,
    My business is to create.

I strongly feel that creation is the best form of comprehension, more so as it applies to physics. I feel that our best students should get involved in
1.9 Grading and assessment

The lab demonstrators and myself are responsible for the overall grading. At the end of the course, students will be ranked into the following categories.

\begin{center}
\begin{tabular}{lc}
A+ & Exceptional \\
A  & Good \\
B  & Satisfactory \\
F  & Fail \\
\end{tabular}
\end{center}

We have prepared an individualized marking sheet for each experiment. The marking premises are largely determined by the "conceptual objectives" mentioned in the lab manual. However, these outcomes have also been quantified, minimizing the possibilities of subjective assessment. For example, if one of the learning outcomes is "understanding how errors propagate", the corresponding quantifiable marking premise would be, "was the student able to find the error in the spring constant, given the uncertainty in the measurement of displacement?" Some sample marking premises for one of our experiments, are reproduced below,

- Has the experimenter calculated the error in the spring constant $k$?
- Was the experimenter able to identify the coordinates in the picture frames?
- What did the experimenter do to avoid the parallax error?
- Does the experimenter understand significance of the semi-log plot and its relative (de)merits?
The overall grade will be assigned after compiling all the eight marking sheets for the student. Note that in order to fail the course, you need to be extremely non-serious or miss out on two or more lab sessions.
Chapter 2

Error Analysis in the Experimental Physics Lab

Sohaib Shamim and Sabieh Anwar

In science, the word 'error' does not mean a mistake. In fact, the term refers to the fact that we cannot make measurements to infinite accuracy and precision and we cannot eliminate them by being very careful. The best we can do is to ensure that errors are as small as reasonably possible and to have a reliable estimate of how large they are.

2.1 How to report and use uncertainties

The correct way to report a reading is to state the measurement and the associated uncertainty. For example, the metre rule in Figure 1 reads 128.9; the associated uncertainty is ±0.1 cm. Thus we write 128.9 ± 0.1 cm. This specifies the most plausible value and the range within which we are confident the quantity lies (between 128.8 and 129.0 cm). Here the error is reported to 0.1 cm because a metre rule has a minimum division of 1 mm (0.1 cm) and we cannot make any trustworthy measurement smaller than 1 mm.

\[
\text{measured value of } x = x_{\text{average}} \pm \delta x
\]
CHAPTER 2. ERROR ANALYSIS IN THE EXPERIMENTAL PHYSICS LAB

Significant Figures

Since $\delta x$ is only an estimate of the uncertainty, therefore we must not quote it with too much precision. For example, writing the value of gravity, $g$ as $g = 9.82 \pm 0.02385 \text{ms}^{-2}$ would be absurd. In very high precision work, uncertainties are reported to, at the most, 2 significant figures but for our purposes we can round off our error to 1 significant figure. Applying this principle, we can rewrite the value of $g$ as,

$$g = 9.82 \pm 0.02 \text{ms}^{-2}.$$ 

Now let’s look at another example. Suppose the speed of a rocket is measured as 6050.78 ms$^{-1}$. Presenting this result with the uncertainty of 30 as 6051.78$\pm$30 is ridiculous. $\pm 30$ implies that the velocity could be as low as 6020 and as high as 6080. This means that the last two numbers 7 and 8 have no significance because we can only accurately measure to the tenth digit and not to any decimal values. Rounding off the measured speed and writing it as 6050 $\pm$ 30 ms$^{-1}$ will make much more sense. If, however, the uncertainty is 0.3, we can write the value as 6050.8$\pm$0.3. If the uncertainty is 3, we can write the result as 6051$\pm$3. The following rule must always be followed.

The last significant figure in any stated answer should usually be of the same order of magnitude (in the same position) as the uncertainty.

A useful practice to reduce inaccuracies is to retain at least 1 significant figure more than what is finally justified, when carrying out calculations. A digital calculator will happily store result to many significant figures and the answer can be rounded off at the end.

2.2 Types of errors

Generally, there are two kinds of errors: systematic and random.

![Figure 2.1: Measuring from a metre scale.](image)
2.2. TYPES OF ERRORS

Systematic errors

These are the result of certain instrument offset or a repeatable measurement error. This will cause a measurement to be always smaller than or greater than the "true value". In some sense, large systematic errors must be eliminated in any good experiment. However, small systematic errors will always be present.

Random errors

These errors are random fluctuations in the measured values that we measure. Random errors are easily identifiable by repeating the experiment. If repeated measurements show the value is changing every time we measure it, then this is the signature of random errors. Figure 2 shows how we can interpret random and systematic errors.

Generally speaking, systematic errors are much harder to detect than random errors and their detection depends on the scientist’s instincts and experience. You will learn to detect systematic errors once you have gained a deeper understanding of the experimental culture and mastered various experimental techniques. We will therefore, focus only on random errors and the techniques by which they can be minimized.

Now consider the following example. We want to measure the time it takes for a ball to travel 10 cm. If we only make one measurement, the answer tells us nothing about the errors involved. Therefore, we repeat the experiment a few times to learn about the mean value. Figure 3(a) shows that repeating the experiment 20 times gives a mean of 5 seconds. If we want to learn more about the mean value and the errors involved, we will have to repeat the experiment a large number of times. So we now increase the repetition number to 100. We

![Diagram](image_url)

Figure 2.2: (a) small random errors; small systematic errors (b) small random errors; large systematic errors (c) large random errors; small systematic errors (d) large random errors; large systematic errors
Figure 2.3: (a) Histogram for repeating the experiment 20 times. (b) Repeating the experiment 100 times gives the resulting histogram a close resemblance to a Gaussian distribution.

see that with increasing repetition number, we obtain a bell shaped curve, known as a Gaussian curve as shown in Figure 3(b). A Gaussian curve has the following characteristic equation

$$
\phi_{\mu,\sigma}(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x - \mu)^2}{2\sigma^2}}
$$

(2.1)

where \( \mu \) is the mean and \( \sigma \) is the standard deviation, a concept we will come across in the next section. The spread of this Gaussian curve determines the standard deviation (\( \sigma \)) of our final result. If there is a large spread, then we have a large standard deviation and the results are not precise. If the spread is small, then we have a small standard deviation which in turn means a precise result. Applying Equation (2.1) to our example with a mean of 5 and standard deviations of 1 (less precise) and 0.5 (more precise), we see that the spread in Figure 4(a) is more than that of Figure 4(b).

Figure 2.4: (a) The spread is large; we have a large random error. (b) The spread is small; small random error
2.3 Quantifying errors in measurements

The statistical method for finding the best value for a measurement is to repeat the measurement many times and then take the average value. Suppose we measure the time for the ball to travel 10 cm. This time, we repeat the measurements 5 times and calculate the average value. The readings are recorded in Table 1.

<table>
<thead>
<tr>
<th>time(s) $x_i$</th>
<th>deviation from average(s) $d_i$</th>
<th>square deviation ($d_i^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>-0.2</td>
<td>0.04</td>
</tr>
<tr>
<td>5.3</td>
<td>0.0</td>
<td>0.00</td>
</tr>
<tr>
<td>5.5</td>
<td>0.2</td>
<td>0.04</td>
</tr>
<tr>
<td>5.1</td>
<td>-0.1</td>
<td>0.04</td>
</tr>
<tr>
<td>5.5</td>
<td>0.2</td>
<td>0.04</td>
</tr>
</tbody>
</table>

average: 5.3s average deviation: 0.0 average square deviation: 0.03

It turns out that the average deviation is zero! For random errors, we are as likely to overestimate a value as underestimate it. A much more useful quantity is the square of the deviation. The sum of the square of deviation is 0.16, a non zero number! We can now take the average of this number and square root it to get the uncertainty. This final answer is known as standard deviation.

Let’s make the following definitions. $X$ is the true value of some observable quantity, for example the true time it takes for the ball to travel 10 cm; $x_i$’s are the individual measurements obtained in the multiple experiments. The error $e_i = x_i - X$ is the difference between a reading and the true value and the deviation $d_i = x_i - \bar{x}$ is the distance from the mean value. The standard errors and standard deviations are defined as follows.

**Standard error in a single measurement**

The square of the standard error is defined as,

$$\sigma^2 = \sum \frac{e_i^2}{n} = \frac{e_1^2 + e_2^2 + e_3^2 + \ldots + e_n^2}{n}.$$

where $n$ is the total number of measurements; in the example we have discussed above, $n = 5$. It is quite obvious that $\sigma$ cannot be directly determined because the true value $X$ and hence the true errors $e_i$’s are unknown.

**Standard error in the mean**
\[ \sigma_m^2 = \frac{\sigma^2}{n}, \text{ or } \sigma_m = \frac{\sigma}{\sqrt{n}} \]  

(2.3)

is the square of the standard error in the mean. As \( n \) increases, one can hope to reduce the uncertainty in the standard error in the mean. This is the most often quoted value in measurements. Of course, we cannot calculate \( \sigma_m \) without \( \sigma \). This seems like a paradoxical problem.

**Standard deviation of the measurements**

A straightforward method, however, is to calculate the standard deviation which utilizes the mean of the readings \( \bar{x} \) which is readily calculable. Writing out the standard deviation in mathematical form,

\[ s^2 = \sum \frac{d_i^2}{n} = \frac{d_1^2 + d_2^2 + d_3^2 + \ldots + d_n^2}{n}. \]  

(2.4)

In our example cited above, \( s^2 = 0.03 \). Once we calculate \( s \), we can use these formulas for determining the standard errors.

\[ \sigma = \sqrt{\frac{n}{n-1} s} \]  

(2.5)

\[ \sigma_m = \frac{\sigma}{\sqrt{n}} = \sqrt{\frac{1}{n-1} s}. \]  

(2.6)

Hence, we have devised a method of calculating the standard errors and standard errors in the means using a directly calculable quantity, the standard deviation \( s \). Note that if \( n \) is large, which is a statistically desirable situation, the calculations based on the standard deviation and the standard error coincide, \( s \approx \sigma \).

Here is a straightforward proof of Equation (2.5).
\[ s^2 = \frac{\Sigma a^2}{n} = \frac{\Sigma (x_i - \bar{x})^2}{n} = \frac{\Sigma (x_i - X + X - \bar{x})^2}{n} = \frac{1}{n} \Sigma (e_i - E)^2 \text{ where } E \text{ is the error in the mean,} \\
= \frac{1}{n} \left[ \Sigma e_i^2 + E^2 - 2\Sigma e_i E \right] = \sigma^2 + E^2 - 2E(E) \text{ identifying the definitions of } \sigma \text{ and } E = \frac{\Sigma e_i}{n} \\
= \sigma^2 - E^2 \\
= \sigma^2 - \sigma_m^2 \\
= \sigma^2 - \sigma^2/n \text{ using the relationship (2.3).} \\
\]

and Equation (2.5) follows.

### 2.4 Checking Relationships with a Graph

![Graph Image](image-url)

Figure 2.5: Repetition of measurements with a fixed set of mass values. For each mass the values of \( T^2 \) form a distribution centered about the true value \( T^2 \).

Many physical laws imply that one quantity is proportional to another. Many experiments in the teaching laboratories are designed to check this kind of proportionality. To test whether a certain quantity, \( y \) is proportional to another variable, \( x \), we can plot a graph of \( y \) against \( x \) and see if the points lie on a straight line. Because a straight line is so easily recognizable, this method is a simple and effective way to check for proportionality. For example, Hooke’s Law states that the extension in the suspended spring is directly proportional to the load applied. A mass on the suspended spring will cause it to perform simple harmonic motion. The time period for one such oscillation is given by

\[ T = 2\pi \sqrt{\frac{m}{k}}. \]
Figure 5 shows a plot of $T^2$ against $m$, giving us a straight line with a gradient $\frac{4\pi^2}{k}$. As we expect, all the points do not lie on a straight line. This is because of random nature of errors. By drawing a line of best fit, we ensure that these errors are minimized.

The Gaussian curves drawn about points $A$, $B$ and $C$ show the probability of obtaining a particular value. If we repeat the readings a very large number of times for a particular mass, we will obtain different values of $T^2$. Plotting these values as in Figure 3, we will see that these $T^2$ values follow the Gaussian distribution. Point $A$ and $C$ lie very close to the fitted line and hence have the highest probability of occurrence (i.e. if we repeat the readings, most of the time we will get values very close to $A$ and $C$). $B$ is the furthest from the most probable result and hence has the lowest probability.

### 2.5 Propagation of Uncertainties

Most physical quantities cannot be measured directly. First, we measure one or more quantities that can be directly measured and then use these quantities to calculate the quantity of interest. For example, the velocity $v$ of a car is measured by measuring the time it takes to travel a particular distance and then calculating the speed by using $v = \frac{d}{t}$. We must first estimate the uncertainty in the measured quantity and then figure out how these uncertainties "propagate" through the calculations to produce an uncertainty in the final deduced answer.

The mathematical treatment for the following results is outside the scope of this course, but has been attached as an appendix for those interested.

#### Uncertainty in Sums and Differences

If $q = x + y$ or $z = x - y$, the uncertainty in $q$ is given by:

$$
\delta q = \sqrt{(\delta x)^2 + (\delta y)^2}
$$

If several quantities $x, \ldots, w$ are measured with uncertainties $\delta x, \ldots, \delta w$, and the measured values used to compute

$$q = x + \cdots + w \text{ or } q = x - \cdots - w,$$
then the uncertainty in the computed value of $q$ is the sum

$$\delta q = \sqrt{(\delta x)^2 + \ldots + (\delta w)^2}$$

i.e. the uncertainties always add, no matter whether we are adding or subtracting the measured quantities.

**Uncertainty in Products and Quotients**

If the equation is $q = xy$ or $q = \frac{x}{y}$, then the uncertainty in $q$ is:

$$\delta q = \sqrt{(\frac{\delta x}{x})^2 + (\frac{\delta y}{y})^2}$$

In general, if several quantities $x, \ldots, w$ are measured with uncertainties $\delta x, \ldots, \delta w$, and the measured values are used to compute

$$q = \frac{x \times \cdots \times y}{u \times \cdots \times w}$$

the uncertainty in $q$ is given by:

$$\delta q = q \sqrt{(\frac{\delta x}{x})^2 + \ldots + (\frac{\delta y}{y})^2 + (\frac{\delta u}{u})^2 + \ldots + (\frac{\delta w}{w})^2}$$

**Uncertainty in a Power**

If the equation is $q = x^m y^n$, the uncertainty is given by:

$$\delta q = q \sqrt{(m \delta x)^2 + \ldots + (n \delta y)^2}$$

If several quantities $x, \ldots, w$ are measured with uncertainties $\delta x, \ldots, \delta w$, and the measured values are used to compute,

$$q = \frac{x^m \times \cdots \times z^n}{u^a \times \cdots \times w^b}$$

the uncertainty is given by:
\[ \delta q = q \sqrt{\left( \frac{m\delta x}{x} \right)^2 + \left( \frac{n\delta z}{z} \right)^2 + \left( \frac{\delta u}{u} \right)^2 + \cdots + \left( \frac{\delta w}{w} \right)^2} \]
APPENDIX

Suppose \( x, y \) are measured with independent and random uncertainties \( \delta x \) and \( \delta y \) and the uncertainties are used to compute the uncertainty in \( q(x, y) \). We make large number of measurements, \( N \), of \( x \) and \( y \) and calculate \( x, y, \sigma_x \) and \( \sigma_y \). Using the Taylor series,

\[
q = q(x, y) \approx q(x, y) + \frac{\partial q}{\partial x}(x - x) + \frac{\partial q}{\partial y}(y - y) \tag{A-1}
\]

with partial derivatives being taken at point \( x=x, y=y \). The mean value \( q \) now becomes

\[
q = \frac{1}{N} \sum_{i=1}^{N} q_i = \frac{1}{N} \sum_{i=1}^{N} \left( q(x, y) + \frac{\partial q}{\partial x}(x_i - x) + \frac{\partial q}{\partial y}(y_i - y) \right) \tag{A-2}
\]

It can be easily shown that \( \sum \frac{\partial q}{\partial x}(x - x) \) and \( \sum \frac{\partial q}{\partial y}(y - y) \) equals zero because any over estimates in \( x \) are counter balanced by under estimates in the measurements and the same is also true for \( y \), and we are left with a very trivial result

\[
q = q(x, y) \tag{A-3}
\]

Now the error in \( q \) is just the standard deviation of \( q \), given by

\[
\sigma_q^2 = \frac{1}{N} \sum (q_i - q)^2. \tag{A-4}
\]

Substituting (A-2) in (A-4),

\[
\sigma_q^2 = \frac{1}{N} \sum \left( \frac{\partial q}{\partial x}(x - x) + \frac{\partial q}{\partial y}(y - y) \right)^2 \tag{A-5}
\]

After a little bit of algebra, we come to the result,

\[
\sigma_q^2 = \sum \left( \frac{\partial q}{\partial x} \frac{\partial q}{\partial y} \right)^2, \tag{A-6}
\]

where

\[
\sigma_{xy} = \frac{1}{N} \sum (x - x)(y - y) \tag{A-7}
\]

We have already shown that \( \sum (x - x) \) and \( \sum (y - y) \) equals zero and hence \( \sigma_{xy} \) also equals zero. Hence, the error in \( q \) is given by:

\[
\sigma_q^2 = \left( \frac{\partial q}{\partial x} \right)^2 \sigma_x^2 + \left( \frac{\partial q}{\partial y} \right)^2 \sigma_y^2 \]
Chapter 3

Questions for Error Analysis

Sabileh Anwar

Q.No.1

A group of students measure $g$, the acceleration due to gravity, with a compound pendulum and obtain the following values in units of m s$^{-2}$.

\[
\begin{array}{ccccccc}
\end{array}
\]

Calculate the mean and the residuals (deviations). Hence estimate $\sigma$. Give the best estimate of $g$, together with its error, for the group.

Q.No.2

In an undergraduate practical class in the Cavendish Laboratory (Cambridge University, UK) there was an experiment, originally devised by Searle, to measure the Young modulus $E$ for steel by applying a known load to a rod and measuring the deflection by an optical method based on Newton’s rings. Although ingenious and capable of considerable precision in the hands of a skilled experimenter, such as Searle himself, the results obtained by the students were found to have a considerable scatter. The experiment was therefore replaced by one in which a horizontal steel beam was supported near its ends, and the deflection when a known load was applied at the centre was measured directly by a dial indicator.

The values obtained for $E$ by the last 10 students who did the Newton’s rings experiment and by the first 10 who did the dial indicator experiment are given
below.

The values are in units of $10^{11}$ N m$^{-2}$.

**Newton's rings experiment**

| 1.90 | 2.28 | 1.74 | 2.27 | 1.67 | 2.01 | 1.60 | 2.18 | 2.18 | 2.00 |

**Dial indicator experiment**

| 2.01 | 2.05 | 2.03 | 2.07 | 2.04 | 2.02 | 2.09 | 2.09 | 2.04 | 2.03 |

For each set of values, calculate the mean value of $E$, and estimate the standard error in the mean. Do the results indicate any systematic difference in the two experimental methods?

**Q.No.3**

In the following examples, $Z$ is a given function of the independently measured quantities $A, B, \ldots$. Calculate the value of $Z$ and its standard error $\Delta Z$ from the given values of $A \pm \Delta A, B \pm \Delta B, \ldots$.

(a) $Z = A^2 \quad A = 25 \pm 1 \quad (A-1)$

(b) $Z = A - 2B \quad A = 100 \pm 3 \, \text{and} \, B = 45 \pm 2 \quad (A-2)$

(c) $Z = \frac{A^2}{B}(C^2 + D^{3/2}) \quad A = 0.100 \pm 0.003$

\hspace{1cm} $B = 1.00 \pm 0.05$

\hspace{1cm} $C = 50.0 \pm 0.5$

\hspace{1cm} $D = 100 \pm 8 \quad (A-3)$

(d) $Z = A \ln B \quad A = 10.00 \pm 0.06 \, \text{and} \, B = 100 \pm (A-4)$

(e) $Z = 1 - \frac{1}{B} \quad A = 50 \pm 2. \quad (A-5)$

**Q.No.4**

A weight $W$ is suspended from the centre of a steel bar which is supported at its ends, and the deflection at the centre is measured by means of a dial height-indicator whose readings are denoted by $y$. The following values are obtained:

Calculate the best value of the slope and the intercept by the method of least squares.
Q. No. 5

The volume $V$ of a rectangular block is determined by measuring the lengths $l_x, l_y, l_z$ of its sides. From the scatter of the measurements, a standard deviation of 0.01% is assigned to each dimension. What is the standard error in $V$, if,

1. the scatter is due to the errors in setting and reading the measuring instrument, and
2. if it is due to temperature fluctuations?
Chapter 4

Solution key for error analysis problems

Amrozia Shaheen and Muhammad Sabieh Anwar

Q.No.1

The mean of the measured values of acceleration due to gravity is,

$$\bar{x} = \frac{\sum x_i}{n} = \frac{9.81 + 9.79 + 9.84 + 9.81 + 9.75 + 9.79 + 9.83}{7} = 9.80.$$

Deviation from the mean value is,

$$d_i = x_i - \bar{x},$$

so, for the measured data, the deviations are,

<table>
<thead>
<tr>
<th>( d_i ) (m s(^{-2}))</th>
<th>( d_i ) (cm s(^{-2}))</th>
<th>((d_i \text{ (cm s}^{-2})^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>-0.01</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0.04</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>0.01</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>-0.05</td>
<td>-5</td>
<td>25</td>
</tr>
<tr>
<td>-0.01</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0.03</td>
<td>3</td>
<td>9</td>
</tr>
</tbody>
</table>
The square of standard deviation, $s$, is,
\[
    s^2 = \frac{\sum d^2}{n} = 7.71 \text{ cm}^2 \text{s}^{-4} = 7.71 \times 10^{-4} \text{ m}^2 \text{s}^{-4}.
\]

implying,
\[
    s = 0.03 \text{ m/s}^2.
\]

The standard error, $\sigma$, can be find out using the following relationship,
\[
    \sigma = \sqrt{\frac{n}{n-1}} s
    = 0.03 \text{ m/s}^2.
\]

As expected the standard error is approximately equal to the standard deviation, $\sigma \approx s$.

The standard error in the mean is,
\[
    \sigma_m = \frac{\sigma}{\sqrt{n}}
    = 0.01 \text{ m/s}^2.
\]

Therefore, the best estimated value of $g$, along with its uncertainty is,
\[
    g = (9.80 \pm 0.01) \text{ m/s}^2.
\]

There is only one significant figure in the uncertainty and the least significant digit of the mean value has the same position as the non-zero digit of the uncertainty.

Q.No.2

Newton's ring experiment

The mean value is,
\[
    \bar{x} = \frac{\sum x_i}{n} = 1.98 \times 10^{11} \text{ N/m}^2.
\]

The deviations are.

The square of standard deviation is,
\[
    s^2 = \frac{\sum d^2}{n}
    = 5.599 \times 10^{20} \text{ N}^2 \text{m}^{-4}.
\]

implying,
\[
    s = 0.24 \times 10^{11} \text{ N/m}^2.
\]

The standard error is,
\[
    \sigma = \sqrt{\frac{n}{n-1}} s
    = 0.25 \times 10^{11} \text{ N/m}^2.
\]
Furthermore, the standard error in the mean is,

\[ \sigma_m = \frac{\sigma}{\sqrt{n}} \]

\[ = 0.08 \times 10^{11} \text{ Nm}^{-2}. \]

The best estimate of \( E \), along with its uncertainty is,

\[ E = (1.98 \pm 0.08) \times 10^{11} \text{ Nm}^{-2}. \]

**Dial indicator experiment**

The mean value is,

\[ \bar{x} = 2.05 \times 10^{11} \text{ Nm}^{-2}. \]

Once again, the deviations are,

The square of standard deviation is,

\[ s^2 = 7.1 \times 10^{18} \text{ N}^2\text{m}^{-4}. \]

gives

\[ s = 0.03 \times 10^{11} \text{ Nm}^{-2}. \]

The standard error is,

\[ \sigma = 0.03 \times 10^{11} \text{ Nm}^{-2}. \]
And the standard error in the mean value is,
\[
\sigma_m = \frac{\sigma}{\sqrt{10}} = 0.01 \times 10^{11} \text{ Nm}^{-2}.
\]
The best estimate of \(E\), together with its error, is,
\[
E = (2.05 \pm 0.01) \times 10^{11} \text{ Nm}^{-2}.
\]
The results of both the experiments are indistinguishable because the ranges overlap, indicating that no systematic errors are present. The dial indicator result is just more precise.

**Q.No.3**

(a) Given value of \(A = 25 \pm 1\), so the value of \(Z\) is,
\[
Z = A^2 = 625.
\]
The standard error in \(Z\) is,
\[
\left(\frac{\Delta Z}{Z}\right) = \left(\frac{2\Delta A}{A}\right)
\]
\[
\Delta Z = 50.
\]

(b) Given \(A = 100 \pm 3\), and \(B = 45 \pm 2\),
\[
\]
And,
\[
\Delta Z = \sqrt{(\Delta A)^2 + (2\Delta B)^2} = 5.
\]

(c) Given \(A = 0.100 \pm 0.003\), and \(B = 1.00 \pm 0.05\), \(C = 50.0 \pm 0.5\), \(D = 100 \pm 8\).
\[
Z = \frac{A}{B} \left( c^2 + D^{3/2} \right) = 350.
\]
2.5. PROPAGATION OF UNCERTAINTIES

We will evaluate \( \Delta Z \) using the following relationships.

Using the above results, we can calculate,

\[
\Delta Z = \sqrt{\left(\frac{C^2 + D^{3/2}}{B}\right) \Delta A^2 + \left(-\frac{A}{B^2}(C^2 + D^{3/2})\Delta B\right)^2 + \left(\frac{2AC}{B} \Delta C\right)^2 + \left(\frac{3A}{2B} D^{1/2} \Delta D\right)^2}
\]

\[
= 24.2.
\]

(d) Given \( A = 10.00 \pm 0.06, B = 100 \pm 2 \).

\[
Z = A \ln B = 46.1.
\]

Hence,

\[
\Delta Z = \sqrt{\left(\frac{dZ}{dA} \Delta A\right)^2 + \left(\frac{dZ}{dB} \Delta B\right)^2}
\]

\[
= \sqrt{(\ln B \Delta A)^2 + \left(\frac{A}{B} \Delta B\right)^2}
\]

\[
\Delta Z = 0.3.
\]

(e) Given \( A = 50 \pm 2 \).

\[
Z = 1 - \frac{1}{A} = 0.98.
\]

Hence,

\[
\left(\frac{\Delta Z}{Z}\right)^2 = \left(\frac{\Delta A}{A}\right)^2.
\]

implying,

\[
\Delta Z = 0.04.
\]

Q.No.4

The mean values of the measured data are,

\[
\bar{x} = 2.25 \text{ kg} \approx 2.3 \text{ kg}
\]

\[
\bar{y} = 865 \mu \text{m}.
\]

The best value of the slope can be deduced using the following relationship,

\[
m = \frac{\sum (x_i - \bar{x})y_i}{\sum (x_i - \bar{x})^2}
\]

\[
= -349.2 \mu \text{m/kg}.
\]
Figure 4.1: Plot of deflection versus weight.

The intercept value is,

\[
    c = y - mx = 1.65 \times 10^3 \approx 1.7 \times 10^3.
\]

Also plot the results as shown in Figure (4.1).

**Q.No.5**

The standard deviation in each dimension is 0.01% \( \approx 10^{-4} \).

The volume of the rectangular block is,

\[
    V = h b l.
\]

(a) For this part, the errors affect the three sides independently. Hence, the standard error in \( V \) will be,

\[
\frac{\Delta V}{V}^2 = \left( \frac{\Delta h}{h} \right)^2 + \left( \frac{\Delta b}{b} \right)^2 + \left( \frac{\Delta l}{l} \right)^2
\]

\[
\frac{\Delta V}{V} = 3 \times 10^{-8}
\]

\[
\frac{\Delta V}{V} = 1.7 \times 10^{-4}
\]

\[
\approx 0.02 \%
\]
(b) For temperature variations, all sides are affected equally. Therefore, one can use the formula for volume with equal lengths,

\[ V = l^3. \]

The error in volume will be,

\[ \left( \frac{\Delta V}{V} \right)^2 = \left( \frac{3 \Delta l}{l} \right)^2 \]

\[ \left( \frac{\Delta V}{V} \right) = \left( \frac{3 \Delta l}{l} \right) \]

\[ = 0.03 \%. \]

This result shows that the overall uncertainty can increase, if the errors are not independent nor random.
Chapter 5

Introduction to Matlab for Experimental Physics

Waqas Mahmood and Sabieh Anwar

Data analysis and representation are vital steps in any experimental exercise. They lend meaning to the experiment and provide insight leading to a more fundamental understanding of the underlying concept. Intelligent data processing and representation also help the experimenter in re-designing the experiment for increased accuracy and precision. Clever thinking may even encourage her to adapt and tailor the procedural steps to elicit some otherwise hidden facet.

In the experimental physics lab, we will use Matlab for,

- analyzing experimental data and computing errors,
- curve fitting, and
- graphically representing experimental data.

The present write-up serves as a first introduction to Matlab. Students who are not familiar with Matlab, or even with the computer, need not to worry. We will proceed slowly, allowing everyone to familiarize and acclimatize with the culture of computing. Luckily, Matlab is a highly user-friendly and interactive package that is very easy to learn. Furthermore, subsequent laboratory sessions will give all of us ample opportunity to practice Matlab.
It is important that every student independently works through all the examples given in this hand-out and attempts all challenge questions. These challenge questions are labelled with the box Q.

**APPROXIMATE PERFORMANCE TIME** 6-8 hours of independent work

This tutorial has been split up into the following sections:

1. Vectors and matrices
2. Graphs and plotting
3. Curve fitting

### 5.1 Vectors and Matrices

**Starting Matlab**

You can start Matlab by double-clicking on the Matlab icon located on the Desktop. The Matlab environment launches showing three windows. On the top left is the *directory window*, showing the contents of the working directory. On the bottom left is the *history window*, displaying your recently executed commands. On the right is the larger-sized *command window*. This is where you will type in your commands and where the output will be displayed.

Now let us get started with the exercise. The simplest calculation is to add two numbers. In the command window, type

\[
2 + 3
\]

What do you see? Indeed, 5, displayed as the answer (*ans*) in the command window. If we terminate the command with the semi-colon,

\[
2 + 3;
\]

the output 5 will not be displayed.

Now take the square of a number, for example, by typing,

\[
5 ^ 2
\]

and verify if you get the correct answer.
Creating Vectors and Matrices

Matlab is centred around the manipulation of matrices. In fact, the word Matlab is acronym for MATrix LABoratory. Let us generate a simple list of numbers, called a vector. This vector comprises all even numbers greater than or equal to 2 and less than 20. We call this vector evenlist.

\[
\gg \text{evenlist} = [2 \ 4 \ 6 \ 8 \ 10 \ 12 \ 14 \ 16 \ 18]
\]

The vector will be displayed, all entries ranging from 2 to 18 lined up in a row. We have just created a row vector. A compact way of creating evenlist would be to write,

\[
\gg \text{evenlist2} = 2:2:18
\]

with the first 2 representing the first element, the second 2 representing the step size and the 18 showing the last element of the vector. Indeed evenlist and evenlist2 are equal. At some later stage, if we want to recall what the vector evenlist2 is, we just retype the label.

\[
\gg \text{evenlist2}
\]

How do we make a column vector, where all the entries are arranged vertically instead of horizontally? We can use the semicolon as a delimiter among rows.

\[
\gg \text{evenlist3} = [2; \ 4; \ 6; \ 8; \ 10; \ 12; \ 14; \ 16; \ 18]
\]

Alternatively, we can avoid keying in the numerical entries by taking the transpose of the row vector evenlist2.

\[
\gg \text{evenlist4} = \text{evenlist3}';
\]

Matrix Arithmetic

Another simple example illustrates matrix multiplication in MATLAB.

\[
\gg a = [2 \ 4 \ 6; \ 1 \ 3 \ 5; \ 7 \ 9 \ 11];
\]

The above operation generates a matrix of order 3 x 3.
\[
\begin{pmatrix}
2 & 4 & 6 \\
1 & 3 & 5 \\
7 & 9 & 11
\end{pmatrix}.
\]
(A-1)

Type in the command,

\[\gg a \cdot 2;\]

This performs the product of the matrices as \(a \cdot a\) and the resulting matrix is,

\[
\begin{pmatrix}
50 & 74 & 98 \\
40 & 58 & 76 \\
100 & 154 & 208
\end{pmatrix}.
\]
(A-2)

Now perform the following operation on the same matrix,

\[\gg a \cdot ^2\]

This operation just takes the square of each entry as shown,

\[
\begin{pmatrix}
4 & 19 & 36 \\
1 & 9 & 25 \\
49 & 81 & 121
\end{pmatrix}.
\]
(A-3)

By typing \(a'\) in the command window, we get the transpose of the generated matrix \(a\) as,

\[
\begin{pmatrix}
2 & 1 & 7 \\
4 & 3 & 9 \\
6 & 5 & 11
\end{pmatrix}.
\]
(A-4)

To understand how MATLAB interprets the forward slash / and the backward slash \, we try some simple commands.

By typing,

\[\gg a=4/2\]

We obtain the answer 2, the result of a division operation. That is, the number on the left hand side of the forward slash is being divided by the number on the
right hand side. On the other hand, if we type,

\[
\gg b = 4/2
\]

The answer is 0.5, which clearly indicates that the number on the right hand side is being divided by the number on the left hand side.

**Introduction to ‘for’ Loops**

For loops are very powerful when we want to continuously update elements of any vector. The typical structure of a for loop is

```
for (condition)  
  statements  
end
```

Now define a row vector,

\[
\gg a = [1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10]
\]

A row vector stores information in the following way,

```
 1 2 3 4 5 6 7 8 9 10
a(1) a(2) a(3) a(4) a(5) a(6) a(7) a(8) a(9) a(10)
```

If we now want to add +1 to all the elements of \( a \), we can write a for loop,

\[
\gg \text{for } k = 1:10 \\
\gg \ a(k) = a(k) + 1;
\gg \text{end}
\]

Matlab now updates every element of \( a \) by +1. The new array will look like,

```
 2 3 4 5 6 7 8 9 10 11
a(1) a(2) a(3) a(4) a(5) a(6) a(7) a(8) a(9) a(10)
```

Note that a **for** statement needs an accompanying **end** statement marking the
5.1. VECTORS AND MATRICES

end of the statements that are being executed.

Extracting Elements from Matrices

Now suppose, we wish to select some entries from a generated row or column vector or from matrices. Define the row vector,

\[ \mathbf{a} = [2 \ 4 \ 6 \ 8 \ 10 \ 12 \ 14 \ 16 \ 18 \ 20] \]

We want to extract the entries from column 3 to 7. We write,

\[ \mathbf{b} = \mathbf{a}(3:7); \]

The colon operator will extract the entries from column 3 to 7, thus giving us the output,

\[ \mathbf{b} = [6 \ 8 \ 10 \ 12 \ 14] \]

Similar procedure can be repeated with a column vector.

We define a matrix by,

\[ \mathbf{a} = [5 \ 8 \ 9; \ 2 \ 4 \ 6; \ 1 \ 3 \ 5] \]

\[
\begin{pmatrix}
5 & 8 & 9 \\
2 & 4 & 6 \\
1 & 3 & 5
\end{pmatrix}
\]  \hspace{1cm} (A-5)

The order of the matrix \( \mathbf{a} \) is \( 3 \times 3 \). Starting from the easiest concept of selecting one single entry from a matrix, we will move on to select the whole row or column of that matrix. Suppose we want to select the entry 4 in the above matrix. We look at the position of that specified entry inside the matrix. The element is located in the second row and second column of the matrix.

\[ \mathbf{b} = \mathbf{a}(2,2) \]

This command takes the value from second row and second column of \( \mathbf{a} \) and saves it in \( \mathbf{b} \). The displayed output is 4.

To select a complete row or column of any matrix we have to use the colon operator, “:” which means that all entries of that specified row or column will
be selected. For example,

\[
\text{\textbackslash r} a(2,:) \\text{\textbackslash n}
\]

displays all the entries of the second row of the matrix, and

\[
\text{\textbackslash r} a(:,2) \\text{\textbackslash n}
\]

displays all the entries in the second column of the matrix \( a \).

If we write \( d=a(:,:) \) in the command window, we get the complete matrix again, i.e., we have selected all the rows and all the columns.

**Higher Dimensional Matrices**

Vectors are one-dimensional arrays, but it is also possible to create arrays or matrices that are two, three or even higher dimensional. For example, let's create a three-dimensional matrix of size \( 3 \times 3 \times 3 \). This means that there are three layers of two-dimensional data. Each layer comprises three rows and three columns. Suppose the first sheet contains the first nine natural numbers arranged in the form of a square.

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9 \\
\end{pmatrix}
\]  
\( \text{(A-6)} \)

The second sheet contains the squares of these numbers,

\[
\begin{pmatrix}
1 & 4 & 9 \\
16 & 25 & 36 \\
49 & 64 & 81 \\
\end{pmatrix}
\]  
\( \text{(A-7)} \)

whereas the third comprises the cubes,

\[
\begin{pmatrix}
1 & 8 & 27 \\
64 & 125 & 216 \\
343 & 512 & 729 \\
\end{pmatrix}
\]  
\( \text{(A-8)} \)

Let's label our tri-layered object as \( F \).

Let us first generate the object \( F \). We pre-allocate some space in the memory by the command,

\[
\text{\textbackslash r} F = \text{zeros}(3,3,3); \\text{\textbackslash n}
\]

Now, in all the three layers we have to initiate the appropriate values. For example,
5.1. VECTORS AND MATRICES

\[ F(:,:,1) = [1 \ 2 \ 3; 4 \ 5 \ 6; 7 \ 8 \ 9] \]

This command will save the first layer of natural numbers in the form of the matrix,

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{pmatrix}
\]

(A-9)

Then the command,

\[ F(:,:,2) = F(:,:,1) \cdot 2 ; \]

generates the squares of the first layer into the second layer. We can view the layer by writing,

\[ F(:,:,2) \]

and the displayed matrix is, indeed,

\[
\begin{pmatrix}
1 & 4 & 9 \\
16 & 25 & 36 \\
49 & 64 & 81
\end{pmatrix}
\]

(A-10)

To generate the cubes from the first layer we type,

\[ F(:,:,3) = F(:,:,1) \cdot 3 \]

To have a look at the generated data we type,

\[ F(:,:,3) \]

yielding,

\[
\begin{pmatrix}
1 & 8 & 27 \\
64 & 125 & 216 \\
343 & 512 & 729
\end{pmatrix}
\]

(A-11)

If we wish to see the matrix element in the second row, third column and in the second layer, we use the command,

\[ a = F(2,3,2) \]

An alternative approach to generate the three dimensional matrix \( F \) is with the
help of a for loop. The programme written below illustrates the use of the for loop.

\>

\>

\>

\>

\>

Yet another alternative approach of creating F, is outlined below. Understand how these options of creating F work.

\>

\>

\>

\>

\>
\[ F(k.m.2) = F(k.m.1)^2; \]
\[ F(k.m.3) = F(k.m.1)^3; \]
\[ \text{end} \]
\[ \text{end} \]

5.2 Graphs and Plotting

Graphs are extremely important in experimental physics. There are three important uses of graphs [1].

- First, with the help of graphs we can easily determine slopes and intercepts.

- Second, they act as visual aids indicating how one quantity varies when the other is changed, often revealing subtle relationships. These visual patterns also tell us if there exist conditions under which simple (linear) relationships break down or sudden transitions take place.

- Third, graphs help compare theoretical predictions with experimentally observed data.

It is customary to plot the independent variable (the "cause") on the horizontal axis and the dependent variable (the "effect") on the vertical axis. In Matlab, the data for the independent and dependent variables are typed in as vectors.

Plotting Basics

Let's consider the seminal experiment [4] performed by Millikan in 1917 for the calculation of the value of Planck's constant \( h \). This experiment, based on the photoelectric effect, also verified Einstein's earlier predictions that light is composed of discrete particles called photons. Millikan's original apparatus as well as our simplified schematic is shown in Figure 5.2.

The experiment works as follows. Monochromatic light (of a fixed wavelength and frequency) falls on a freshly cut surface of sodium metal attached to the electrode \( P \). As a result electrons are ejected from the metal surface and because of their ejection momentum, they cruise their way to the electrode \( Q \). These electrons
constitute a photocurrent that is measured by the ammeter. But this motion is opposed by a voltage that makes Q more negative than P. As Q becomes more and more negative, fewer electrons reach the electrode and the current diminishes. At a certain potential difference, called the stopping voltage \( V_s \), the current finally approaches zero. Millikan repeated the experiment for various light sources. One such set of his readings is listed in Table 5.1.

<table>
<thead>
<tr>
<th>Stopping voltage ( V_s ) (V)</th>
<th>-2.100</th>
<th>-1.524</th>
<th>-1.367</th>
<th>-0.9478</th>
<th>-0.3718</th>
<th>+0.3720</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wavelength ( \lambda ) (Å)</td>
<td>5466</td>
<td>4339</td>
<td>4047</td>
<td>3650</td>
<td>3126</td>
<td>2535</td>
</tr>
</tbody>
</table>

Table 5.1: Millikan's readings for the stopping voltage as a function of the wavelength of the incident light; results extracted from [4].

Now let's plot \( V_s \) as a function of the frequency \( f \), keeping \( V_s \) on the vertical and \( f \) on the horizontal axis. The first step is to input the data in the form of vectors.

\[
\gg \text{wavelength} = [5466 \ 4339 \ 4047 \ 3650 \ 3126 \ 2535] \times 10^{-10};
\]

\[
\gg \text{vs} = [-2.1 \ -1.524 \ -1.367 \ -0.9478 \ -0.3718 \ 0.3720];
\]

Next, we convert the wavelengths to frequencies.

\[
\gg c = 3e8;
\]

\[
\gg f = c./\text{wavelength};
\]

Here \( c \) is the speed of light. \( \gg c=3e8; \) is a compact way of writing \( 3 \times 10^8 \). Also
note the pointwise division of the speed of light by the wavelength, using the familiar ":" operator. The graph is achieved using the command,

```matlab
figure; plot(f,vs);
```

and the horizontal and vertical axes are labelled using,

```matlab
xlabel('frequency f (Hz)');
ylabel('stopping voltage Vs (V)');
```

The resulting graph is shown in Figure 5.3(a). The plot is a solid black line joining the individual data points, even though the points themselves are not distinguished. These points can in fact be highlighted using symbols such as "o", "+" and "*". The colours can also be adjusted. For example, to plot a solid red-coloured line with circles for the data points, we use the command,

```matlab
figure; plot(f,vs,'-o');
```

![Graphs](image)

Figure 5.3: (a) Output from `figure; plot(f,vs);` a solid jagged line connects the data points; (b) output from `figure; plot(f,vs,'-o');` a solid red line connects the data points that are now highlighted; (b) output from `figure; plot(f,vs,'ro');` showing just the data points.
Furthermore, if it is required to display the data points only, suppressing the line that connects between these points, we type,

\[ \texttt{>> figure; plot(f,vs,'r-')}; \]

This latter plot, shown in Fig. 5.3(c) in fact, represents a more justifiable picture of the experimental data. This is because the lines drawn in (a) and (b) represent more than what the data warrants: the lines show that the frequency and the stopping voltage have some kind of jagged relationship, something that is highly likely. A more reasonable prediction is that the relationship is a straight line. In the next section, we will discuss how to draw one such line, using the procedure of least squares curve fitting.

\section*{Q 1.} The deflection of a cantilever beam is the distance its end moves in response to a force applied at the end as shown in Figure 5.4. The following table gives the deflection \( x \) that was produced in a particular beam by the given applied force \( f \). Find a functional relationship between \( x \) and \( f \) and plot the graph.

<table>
<thead>
<tr>
<th>Force ( f ) (Pounds)</th>
<th>0</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>400</th>
<th>500</th>
<th>600</th>
<th>700</th>
<th>800</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deflection ( x ) (inches)</td>
<td>0</td>
<td>0.09</td>
<td>0.18</td>
<td>0.28</td>
<td>0.37</td>
<td>0.46</td>
<td>0.55</td>
<td>0.65</td>
<td>0.74</td>
</tr>
</tbody>
</table>

Table 5.2: An experiment to measure force and deflection in cantilever beam.

\section*{Q 2.} Draw a graph of the function,

\[ y = \frac{\sin t}{t} \quad (A-12) \]

for \( 0 \leq t \leq 10 \).

\section*{Q 3.} For the values of \( x \), \( 0 \leq 2\pi \), show by drawing a graph that,

\[ \sin^2 x + \cos^2 x = 1. \quad (A-13) \]

\section*{Q 4.} Draw a graph of the function,

\[ z = \exp(-0.5t) \cos(20t - 6) \quad (A-14) \]

for \( 0 \leq t \leq 8 \).

\section*{Q 5.} Draw a graph of the function,

\[ y = -x \exp(-x) \quad (A-15) \]

for \( 0 \leq x \leq 10 \).
Q 6. Biomedical engineers often design instrumentation to measure physiological processes, such as blood pressure. To do this, they must develop mathematical models of the process. The following equation is a specific case of one model used to describe the blood pressure in the aorta during systole (the period following the closure of the heart's aortic valve). The variable \( t \) represents time in seconds and the dimensionless variable \( y \) represents the pressure the aortic valve, normalized by a constant reference pressure.

\[
y(t) = e^{-8t} \sin(9.7t + \frac{\pi}{2}). \tag{A-16}
\]

Plot this function for,

\[
t \geq 0. \tag{A-17}
\]
Overlaying Multiple Plots

It is also possible to plot multiple curves on the same figure. This is a highly useful feature as you will soon realize. Consider for example, the load-line analysis of electrical circuits. A voltage source $V_1$, having an internal resistance $R_1$ is connected to the load as shown in the Figure 5.5. This power supply produces a fixed voltage supplying current $i_1$ to the load resulting in a potential drop $V_2$ across the load.

![Figure 5.5: A power supply with resistance $R_1$ and the load are shown in a circuit.](image)

An experimenter built the circuit shown in Figure 5.5. The current-voltage relationship approximated from the experiment was,

$$i_1 = 0.16(e^{0.12V_2} - 1).$$  \hspace{1cm} (A-18)

Let's suppose that we have a supply voltage of $V_1=15$ V and the resistance of the supply is 30 Ohms. In the first step, we write the equation of the circuit using Kirchoff's Voltage Law.

$$V_1 = i_1 R_1 + V_2.$$  \hspace{1cm} (A-19)

which implies,

$$V_1 - i_1 R_1 - V_2 = 0.$$  \hspace{1cm} (A-20)

Load line tells us how the current across the load changes as the voltage across the load is changed. So, we write Equation A-20 in terms of current as,

$$i_1 = \frac{1}{R_1} V_2 + \frac{V_1}{R_1}.$$  \hspace{1cm} (A-21)
which can be re-written after using the values as,

\[ i_1 = -\frac{1}{30} V_2 + 0.5. \] (A-22)

From equations A-18 and A-22, it is difficult to calculate the values of \( i_1 \) and \( V_2 \) because of the exponential factor present in equation A-18. But we can plot the two curves individually and then overlap to find the solution.

We will use Matlab to plot the load voltage \( V_2 \) against the experimentally obtained relation of current and the relation for current obtained from the rearrangement of Kirchoff’s Voltage Law. The point at which these two curves intersect gives us the solution.

\[ V2=0:0.01:20; \] (creating the voltage vector)

\[ \text{expcurrent}=0.16*(\exp(0.12*V2)-1); \] (calculating the current from the experimental relation)

For the calculation of the values of current from the Equation A-22, we write,

\[ \text{thcurrent}= -(1/30)*V2 +0.5; \] (calculating the current from Equation A-22)

The load voltage can then be plotted against current by using the command,

\[ \text{figure; plot(V2,expcurrent,.);} \]

where we have used "." for dots. The output is shown in the Figure 5.6 (a).

To plot the load voltage versus current using Equation A-22, write,

\[ \text{figure; plot(V2,thcurrent,--);} \]

where we have used two hyphens for a dashed line. The output of the above mentioned command is shown in Figure 5.6 (b).

To plot both the graphs simultaneously, one on top of another, write,

\[ \text{figure; plot(V2,expcurrent,.,V2,thcurrent,--);} \]

The result is shown is Figure 5.7. To get the point of intersection press the Data cursor button in the Figure window and then click at the point of intersection. The value shown is the solution.
Figure 5.6: (a) The load voltage $V_2$ and the Experimental Current. (b) The load voltage $V_2$ and the Theoretical Current.

Figure 5.7: Overlaying of two plots and finding the point of intersection.

**Q 7.** Plot the two curves

$$y = \cos x, \quad \text{and} \quad (A-23)$$

$$y = x \quad (A-24)$$

over the range $x \in [0, 3]$ and use the curves to find the solution of the equation $x = \cos x$. 
5.2. GRAPHS AND PLOTTING

Q 8. Plot the two curves

\[ y = 2 \cos x, \quad \text{and} \quad (A-25) \]
\[ y = 2 \sin x \quad (A-26) \]

over the range \( x \in [0, 4\pi] \).

Q 9. Suppose the relationship between the dependent variable \( y \) and the independent variable \( x \) is given by,

\[ y = ae^{-x} + b \quad (A-27) \]

where \( a \) and \( b \) are constants. Sketch a curve of \( y \) versus \( x \) using arbitrary values of \( a \) and \( b \). Is it possible to obtain a straight line that represents this functional relationship?

Resolution of the Graph

Figure 5.8(a) shows the result of plotting a sine curve

\[ \sin(t) \quad (A-28) \]

sampled at intervals of 1 s for a duration of 10 s. As such there are eleven data points contained within the sampled duration. We know from experience that a plot of the sine function should be smooth, unlike the irregular curve shown. Why is there this discrepancy? The reason is that we have not sampled enough points. Decreasing the sampling interval to 0.1 s and hence, increasing the number of samples to 101, we recover a smooth sine curve, shown in Figure 5.8(b). These plots have been made using the following commands.

\[ \triangleright t1=0:1:10; \]
\[ \triangleright x1=\sin(t1); \]
\[ \triangleright \text{figure; plot(t1,x1,'g-o'); \quad (for the subfigure (a))} \]
\[ \triangleright t2=0:1:10; \]
\[ \triangleright x2=\sin(t2); \]
\[ \triangleright \text{figure; plot(t2,x2,'b-v'); \quad (for the subfigure (b))} \]

However, these plots cannot be overlaid one on top of each other using the command \texttt{figure; plot(t1,x1,’g-o’,t2,x2,’b-v’)}; as \texttt{t1} and \texttt{t2} are essentially different vectors. A way around this is to use the following set of commands.
Figure 5.8: (a) Output from `figure; plot(t1,x1,'g-o');` lower resolution graph; (b) output from `figure; plot(t2,x2,'b-v');` higher resolution graph; (b) output from `figure; plot(t1,x1,'g-o'); hold on; plot(t2,x2,'b-v');` whereby the two graphs have been drawn on top of each other.

```matlab
≫ figure; plot(t1,x1,'g-o'); hold on; plot(t2,x2,'b-v');
```

We can also specify the color and size of lines which we use while making a plot. Consider an equation,

\[ y = \tan(\sin(x)) - \sin(\tan(x)) \]  
(A-29)

We plot Equation A-29 for the range \( x \in [-\pi, \pi] \) by writing.

```matlab
≫ x=-pi:pi/10:pi;
≫ y=tan(sin(x))-sin(tan(x));
≫ figure; plot(x,y,'-rs','LineWidth',2,'MarkerEdgeColor','k','MarkerFaceColor','g','MarkerSize',10)
```
Figure 5.9: Illustration of Color and Size of the lines.

This working produces a graph as shown in Figure 5.9 with,

- a dashed line having red color and square markers,
- squares having black color at the edges,
- squares filled in with green color,
- marker size set to 10.

Lines with different styles like solid, dashed and dotted etc. can be drawn with marker types $\times$, $\ast$, $+$ and $o$. The color options include cyan, magenta, yellow, black, red, green, blue and white with symbols $c$, $m$, $y$, $k$, $r$, $g$, $b$, $w$. There are plenty of equations in Matlab for you to explore.

### 5.3 Curve Fitting

Consider, once again, Millikan’s famous experiment for determining the Planck constant. Observe Figure 5.3(c). Can we draw a straight line through these points, not necessarily touching them? What could be the significance of such a line? In the present section, we will explore answers to this question.

#### Linear Relationships

Figure 5.10(a) is a reproduction of the data points shown in Figure 5.3(c). However, in this graph we have also drawn two straight lines. Why straight lines?
Figure 5.10: (a) Data points from Millikan’s experiment [4] with two possible lines defining the functional relationship between $f$ and $V_s$; (b) magnified region from the graph (a), closely showing the data points and the straight lines.

Linear relationships occur naturally in numerous natural instances and that is why they have become the scientist’s favourite. Linear relationships are direct manifestations of direct proportionality. If the variables $x$ and $y$ are directly proportional ($x \propto y$), an equal increase in $x$ always results in an equal increase in $y$. Be it the extension of a spring when loaded with masses, the acceleration of an object as it experiences a force or the magnetic field that winds around a current carrying conductor, linear relationships are ubiquitous. When these linear functions are drawn on paper (or on the computer screen), they become straight lines.

The straight lines we have drawn in Figure 5.10(a) represent a kind of interpolation. In the real experiment, we measure the variables, $(x, y)$. In our case these are frequency and stopping voltage. In a set of measurements, we have six pairs of data points $(x_1, y_1), (x_2, y_2), \ldots, (x_6, y_6)$. What if we want to determine the stopping voltage for a frequency that was not used by Millikan? We could either repeat his experiment with a light source with the desired frequency or estimate using available data. In the latter case, we draw a straight line around the available measurements $(x, y)$. This line negotiates data points not available to the

---

$F = ma$

(Newton’s law)

$F = -kx$

(Hooke’s law)

$B = \mu_0 NI$

(Ampere’s law)
5.3. CURVE FITTING

experimenter.

But what straight line do we actually draw? This is a matter of choice. For example, we have drawn two lines in the Figure. The light colored line takes the first and the last data points as reference and connects these points; whereas the dark colored line connects the mean (or the centre of gravity of the data) to the end point. Both lines are different and at the outset, are equally suitable for defining the linear relationship between the variables of interest.

Let’s briefly digress to see how we plotted, say, the red line. To plot a line, we need an equation for the line. Given two points \((x_1, y_1)\) and \((x_6, y_6)\), a straight line through these will be given by,

\[
\frac{y - y_1}{y_6 - y_1} = \frac{x - x_1}{x_6 - x_1}.
\]  

(A-30)

and in our case \((x_1, y_1) = (0.5488 \times 10^{15}, -2.1)\) and \((x_6, y_6) = (1.1834 \times 10^{15}, 0.3720)\). (These numbers have been taken from the row vectors \(f\) and \(vs\).) After some basic arithmetic (also done in Matlab) we arrive at the following equation for the red line,

\[
y = 3.895 \times 10^{-15}x - 4.2375,
\]  

(A-31)

where in our particular case \(y\) is the stopping voltage \(v_s\) and \(x\) is the frequency \(f\). Similarly, the equation for the blue line was computed by first calculating the means of the \(x\) and \(y\) values. The resulting equation is,

\[
y = 3.895 \times 10^{-15}x - 4.2018,
\]  

(A-32)

yielding a line parallel to the first, but displaced upwards. Figure 5.10(b) shows a close-up of (a), revealing that these lines do not actually touch a majority of the data points, they just graze within that region.

The graph has been plotted by using the following set of commands.

\[
\gg \text{line1=3.895e-15*f-4.2375;}
\]

\[
\gg \text{line2=3.895e-15*f-4.2018;}
\]

\[
\gg \text{figure; plot(f,vs,'ro',f,line1,'g-',f,line2,'b-');}
\]

Least Squares Curve Fitting of Linear Data

Consider Figure 5.11 where a straight line has been drawn around a set of experimentally measured data points \((x, y_i)\). In this example we have \(N = 7\) pairs of
measurements. The line is represented by the equation,

\[ y = mx + c \quad \text{(A-33)} \]

where \( m \) is the \textit{slope} and \( c \) is the \textit{intercept}. Of the many lines that can be drawn, this particular line has a special property that we now investigate. If the reading along the abscissa (\( x \) axis) is \( x_i \), the corresponding measurement along the ordinate (\( y \) axis) is \( y_i \), but the line we have just drawn takes up the value, \( mx_i + c \) instead, which in general, is different from \( y_i \). This difference \[ d_i = y_i - mx_i - c \quad \text{(A-34)} \]

is called the \textit{residual or deviation}. The special line we have drawn has the property that it minimizes the sum of the squares of the deviations,

\[ S = \sum_{i=1}^{N} d_i^2 = \sum_{i=1}^{N} (y_i - mx_i - c)^2. \quad \text{(A-35)} \]

and hence the name \textit{least squares curve fit}. If the \( d_i \)'s are considered to be the errors, the least squares curve fit is the best fit in the sense that it minimizes the squares of the errors.

**Q 10.** Why do we minimize the sum squares of the residuals \( \sum_{i=1}^{N} d_i^2 \) instead of the sum of the residuals \( \sum_{i=1}^{N} d_i \)?

There is an algorithmic procedure for deriving the equation for the least squares fit. The goal is to find the parameters \( m \) and \( c \) that minimize the quantity \( S \). The minimum of \( S \) can be determined from elementary calculus. Take the derivative of \( S \), first with respect to \( m \) and then with respect to \( c \) and put the derivatives equal to zero.
\[
\frac{\partial S}{\partial m} = -2 \sum_{i=1}^{N} x_i (y_i - mx_i - c) = 0 \quad (A-36)
\]

\[
\frac{\partial S}{\partial c} = -2 \sum_{i=1}^{N} (y_i - mx_i - c) = 0. \quad (A-37)
\]

Rearranging Equation A-37, we obtain,
\[
\sum_{i=1}^{N} (y_i - mx_i - c) = 0
\]
\[
\sum_{i=1}^{N} y_i - m \sum_{i=1}^{N} x_i - cN = 0
\]

\[
\Rightarrow c = \frac{\sum_{i=1}^{N} y_i - m \sum_{i=1}^{N} x_i}{N}.
\quad (A-38)
\]

where \( \sum_{i=1}^{N} = \sum_{i=1}^{N} \).

The expression for \( c \) is inserted into Equation A-36 and after some algebraic manipulation,
\[
\sum_{i=1}^{N} x_i (y_i - mx_i - c) = 0
\]
\[
\sum_{i=1}^{N} (x_i y_i) - m \sum_{i=1}^{N} x_i^2 - c \sum_{i=1}^{N} x_i = 0
\]
\[
\sum_{i=1}^{N} (x_i y_i) - m \sum_{i=1}^{N} x_i^2 - \left[ \frac{\sum_{i=1}^{N} y_i - m \sum_{i=1}^{N} x_i}{N} \right] \sum_{i=1}^{N} x_i = 0
\]
\[
\sum_{i=1}^{N} (x_i y_i) - m \sum_{i=1}^{N} x_i^2 - \frac{1}{N} (\sum_{i=1}^{N} x_i) (\sum_{i=1}^{N} y_i) + m \frac{N}{N} (\sum_{i=1}^{N} x_i)^2 = 0.
\quad (A-40)
\]

the following expression for \( m \) pops out,
\[
\frac{\sum_{i=1}^{N} x_i y_i - \frac{1}{N} (\sum_{i=1}^{N} x_i) (\sum_{i=1}^{N} y_i)}{(\sum_{i=1}^{N} x_i)^2 - \left( \frac{\sum_{i=1}^{N} x_i}{N} \right)^2}.
\quad (A-41)
\]

This cumbersome looking expression can be simplified by noticing that,
\[
\frac{\sum_{i=1}^{N} x_i}{N} = \bar{x}
\quad (A-42)
\]
is the mean of \( x_i \) and
\[
\frac{\sum_{i=1}^{N} y_i}{N} = \bar{y}
\quad (A-43)
\]
is the mean of \( y_i \), yielding,
\[
m = \frac{\sum_{i=1}^{N} x_i y_i - N \bar{x} \bar{y}}{\sum_{i=1}^{N} x_i^2 - N \bar{x}^2}.
\quad (A-44)
Furthermore, we can also make use of the following simplifications for the numerator and denominator of the above expression,

\[
\sum_{i=1}^{N} (x_i y_i) - N \bar{y} \quad = \quad \sum_{i=1}^{N} (x_i y_i) - \left( \sum_{i=1}^{N} y_i \right) \bar{x} \\
= \quad \sum_{i=1}^{N} y_i (x_i - \bar{x}), \quad \text{and}
\]

\[
\sum_{i=1}^{N} x_i^2 - N \bar{x}^2 \\
= \quad \sum_{i=1}^{N} x_i^2 - N \bar{x}^2 - 2N \bar{x} \sum_{i=1}^{N} x_i \\
= \quad \sum_{i=1}^{N} x_i^2 + N \bar{x}^2 - 2N \sum_{i=1}^{N} x_i \\
= \quad \sum_{i=1}^{N} (x_i^2 + \bar{x}^2 - 2\bar{x} x_i) \\
= \quad \sum_{i=1}^{N} (x_i - \bar{x})^2.
\]

This tedious but fruitful exercise yields the following compact expression for the slope of the least squares curve fit,

\[
m = \frac{\sum_{i=1}^{N} y_i (x_i - \bar{x})}{\sum_{i=1}^{N} (x_i - \bar{x})^2}.
\]

Substituting the expression for \( m \) back into (A-39) we can determine the intercept,

\[
c = \bar{y} - m \bar{x}.
\]

Q 11. Prove that the least squares curve fit passes through the centre of gravity \((\bar{x}, \bar{y})\) of the measured data.

Now we use Matlab to find the least squares curve for Millikan’s experimental data. The commands that generate the best fit line are given below.

\[
\gg \text{numerator} = \text{sum}((\text{vs}-\text{mean(f)})) ;
\]

\[
\gg \text{denominator} = \text{sum}((\text{f-mean(f)}) .^ \text{2}) ;
\]

\[
\gg m = \text{numerator} / \text{denominator} ;
\]

\[
\gg c = \text{mean(vs)} - m * \text{mean(f)} ;
\]

The values are \( m = 3.9588 \times 10^{-15} \) V/Hz and \( c = -4.2535 \) V. We can now easily plot the least squares fit, shown in Figure 5.12.

\[
\gg \text{line3} = m * f + c ;
\]
5.3. **CURVE FITTING**

> figure; plot(f,vs,'ro',f3,line3,'g-')

![Graph showing the relationship between Vs and f/Hz.](image)

**Figure 5.12:** Data points for Millikan's experiment and the least squares curve fit.

The straight line, in fact, has real physical value as well. For example, according to Einstein's interpretation of the photoelectric effect, light is carried in the form of small packets called photons. Corresponding to the frequency $f$, the photon carries an energy $hf$, where $h$ is Planck's constant. As light is shone on the metal surface, a fraction of the energy called the work function $W < hf$ is absorbed by the metal surface. The ejected electron carries the energy difference $hf - W$ appearing as its kinetic energy. As the voltage $V_s$ is made more and more negative, the number of electrons reaching electrode $Q$ diminishes with only the more energetic electrons being able to overcome the opposing voltage. At the stopping voltage, the maximum kinetic energy equals the voltage barrier. Given a potential of $V_s$, the corresponding potential energy is $eV_s$, $e$ being the charge of the electron. This description allows us to write the following equation,

$$
e V_s = hf - W$$

$$V_s = \left(\frac{h}{e}\right)f - \left(\frac{W}{e}\right). \quad (A-49)$$

Comparing this with the least squares fitted equation A-33, we immediately recognize that the slope $m$ is in fact an estimate of $h/e$ and the intercept $c$ is an estimate of $W/e$. Using the slope and intercept from the best-fit and a value of
$e = 1.6022 \times 10^{-19}$ C, the Planck constant calculates to $h = 6.342 \times 10^{-34}$ Js and the work function to $W = 6.814 \times 10^{-19}$ J or 4.2535 eV.

**Least Squares Curve Fitting of Nonlinear Data**

The concept of curve fitting can also be applied to the *nonlinear* data. Suppose we route a sinusoidal ac voltage through a data acquisition system bringing it into the computer. The hardware samples the voltage, acquiring one sample every 50 ms and saves the first 21 points. The time sampling information is stored in the form of the row vector \( t \) where 0.5 s shows the separation between two sample points.

\[ t=0:0.05:1; \]

The voltage measurements made by the acquisition software are given by another row vector \( v \).

\[ v=[5.4792 \quad 7.4488 \quad 7.5311 \quad 5.7060 \quad 2.4202 \quad -1.5217 \quad -5.1546 \quad -7.5890 \quad -8.2290 \quad -6.9178 \quad -3.9765 \quad -0.1252 \quad 3.6932 \quad 6.5438 \quad 7.7287 \quad 6.9577 \quad 4.4196 \quad 0.7359 \quad -3.1915 \quad -6.4012 \quad -8.1072]; \]

Note that \( \text{size}(t)=\text{size}(v) \). We are asked to fit this data to a least squares curve, a sinusoidal function. Our best fit will be of the form,

\[ A \sin(\omega t + \phi), \]

where \( A \) is the amplitude, \( \omega \) is the angular frequency and \( \phi \) is the phase. The curve fitting procedure determines approximations to these parameters, \( A \), \( \omega \) and \( \phi \); however, the simple algorithm outlined above for linear fits does not work here.

Instead we use the inbuilt Matlab command `lsqcurvefit`. We first make a new function file named `sinusoid.m` that contains the fitting function. Follow the following steps to make a new function file, also called an ‘m-file’.

1. From the **File** menu item, click **New** and **M-file**. A blank text editor opens.

2. Type in the following text in the editor window.

   ```matlab
   function Fout=sinusoid(p,Fin)
   Fout=p(1)*(sin(p(2)*Fin+p(3)));
   end
   ```

   and save the file in the working directory as `sinusoid.m`. 


Let's parse this file, line by line. The first line starts with the label \texttt{function} indicating that this m-file is a function file, or in other words, this file contains the declaration of a function named \texttt{sinusoid} that can be called from inside the command window. The function \texttt{sinusoid} takes in two vector arguments, \( \mathbf{p} \) and \( \mathbf{Fin} \). The former is a vector containing the unknown parameters. In our case \( \mathbf{p} \) has three elements \( p(1) \), \( p(2) \) and \( p(3) \) which are respectively \( A \), \( \omega \) and \( \phi \). The latter \( \mathbf{Fin} \) is the input vector, in our case this is the vector containing the time values. The second line defines the fitting function; this is the Matlab way of writing Equation (A-50). Finally, the m-file ends with the statement \texttt{end}.

Once the fitting function has been defined, we can find the least squares curve\(^2\) using the command,

\[
\gg \text{lsqcurvefit(@sinusoid,[8 10 0],t,v)}
\]

The first argument references the function we have just created. The second argument is a vector containing initial guesses of the unknown parameters. It will be easier for Matlab if we could make intelligent guesses of these parameters. The last two arguments, \( \mathbf{t} \) and \( \mathbf{v} \) are the abscissa and ordinate variables. Matlab returns the values of the parameters, \( A = 7.9551, \omega = 10.0256, \phi = 0.7971 \).

The initial data points and the higher resolution curve fit are then plotted using the set of commands given below.

\[
\gg t2=0:0.005:1; \quad \text{(high sampling rate for plotting the fitted curve)}
\]

\[
\gg \text{cfit=7.9551*sin(10.0256*t2+0.7971);}
\]

\[
\gg \text{figure; plot(t,v,'ro'); hold on;}
\]

\[
\gg \text{plot(t2,cfit,'g-');}
\]

The results are shown in Figure 5.13.

The command,

\[
\gg [x,\text{resnorm}]=\text{lsqcurvefit(@sinusoid,[8 10 0],t,v)}
\]

also returns the sum of the squares of the residuals,

\[
\sum_{i=1}^{N} e_i^2 \quad \text{(A-51)}
\]

which is a measure of the goodness of the fit. Note that \texttt{lsqcurvefit} will also work

\(^2\text{\texttt{lsqcurvefit} requires the optimization toolbox}\)
Figure 5.13: Acquired voltage samples. The measurements are plotted as circles whereas the least squares curve fit is drawn as a solid line.

for linear curve fitting.

Q 12. Suppose a rocket is fired into the space from rest. The distance covered (in miles) by the rocket and the height gained (in miles) is given in the table below.

<table>
<thead>
<tr>
<th>Distance (miles)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (miles)</td>
<td>0</td>
<td>0.53</td>
<td>0.75</td>
<td>0.92</td>
<td>1.07</td>
<td>1.20</td>
<td>1.31</td>
<td>1.41</td>
<td>1.51</td>
<td>1.60</td>
<td>1.69</td>
<td>1.77</td>
<td>1.85</td>
</tr>
</tbody>
</table>

Table 5.3: Height of a rocket versus Distance.

Plot the graph of distance against height and perform curve fitting using an equation,

\[ y = a\sqrt{bx} \]  \hspace{1cm} (A-52)

Q 13. An object covers a distance \( d \) in time \( t \). A measurement of \( d \) with respect to \( t \) produces the set of values given in Table 5.4 [5].

<table>
<thead>
<tr>
<th>( t ) (s)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d ) (m)</td>
<td>0.20</td>
<td>0.43</td>
<td>0.81</td>
<td>1.57</td>
<td>2.43</td>
<td>3.81</td>
<td>4.80</td>
<td>6.39</td>
</tr>
</tbody>
</table>

Table 5.4: Measurements of distance as a function of time.

Plot the distance with respect to \( t \). Then plot with respect to \( t^2 \). If the object was initially at rest, calculate the acceleration. Use curve fitting.
Q 14. Biomedical instruments are used to measure many quantities such as body temperature, blood oxygen level, heart rate and so on. Engineers developing these devices often need a response curve that describes how fast the instrument can make measurements. The response voltage $v$ can be described by one of these equations,

$$v(t) = a_1 + a_2 e^{-3t/T}$$
$$v(t) = a_1 + a_2 e^{-3t/T} + a_3 t e^{-3t/T}$$

(A-53)

where $t$ is the time and $T$ is an unknown constant. The data given in Table 5.5 gives the voltage $v$ of a certain device as a function of time. Which of the above functions is a better description of the data [7]?

<table>
<thead>
<tr>
<th>$t$ (s)</th>
<th>0</th>
<th>0.3</th>
<th>0.8</th>
<th>1.1</th>
<th>1.6</th>
<th>2.3</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v$ (V)</td>
<td>0</td>
<td>0.6</td>
<td>1.28</td>
<td>1.5</td>
<td>1.7</td>
<td>1.75</td>
<td>1.8</td>
</tr>
</tbody>
</table>

Table 5.5: Response of a biomedical instrument switched on at time $t = 0$.

Q 15. In an $RC$ series circuit, a parallel plate capacitor having capacitance $C$ charges through a resistor $R$. During the charging of capacitor, charge $Q$ starts to accumulate on the plates of the capacitor. The expression for growth of charge $V$ is given by,

$$V = V_0 (1 - \exp(-t/\tau))$$

(A-54)

where the time constant $\tau = RC$. Fit the given data in Table 5.7 to the equation for the voltage increase and find the value of $\tau$.

<table>
<thead>
<tr>
<th>$t$ (s)</th>
<th>0</th>
<th>3</th>
<th>6</th>
<th>9</th>
<th>12</th>
<th>15</th>
<th>18</th>
<th>21</th>
<th>24</th>
<th>27</th>
<th>30</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V$ (V)</td>
<td>0</td>
<td>6.55</td>
<td>10</td>
<td>13</td>
<td>14.5</td>
<td>15</td>
<td>16</td>
<td>16.2</td>
<td>16.3</td>
<td>16.5</td>
<td>16.55</td>
</tr>
</tbody>
</table>

Table 5.6: Charging pattern for a capacitor in an $RC$ circuit.

Q 16. When a constant voltage was applied to a certain motor initially at rest, its rotational speed $S(t)$ versus time was measured. The table given below shows the values of speed against time.

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Speed (rpm)</td>
<td>1210</td>
<td>1866</td>
<td>2301</td>
<td>2564</td>
<td>2724</td>
<td>2881</td>
<td>2879</td>
<td>2915</td>
<td>3010</td>
</tr>
</tbody>
</table>

Table 5.7: Motor speed when it is given a push.

Try to fit the given data with the function given below. Calculate the constants
b and c.

\[ S(t) = b(1 - e^{ct}) \]  

(A-55)

Q 17. A hot wire anemometer is a device for measuring flow velocity, by measuring the cooling effect of the flow on the resistance of a hot wire. The following data points are obtained in a calibration test.

![Hot wire anemometer](image)

Figure 5.14: Hot wire anemometer.

<table>
<thead>
<tr>
<th>u (ft/s)</th>
<th>66.77</th>
<th>59.16</th>
<th>54.45</th>
<th>47.21</th>
<th>42.75</th>
<th>32.71</th>
<th>25.43</th>
<th>8.18</th>
</tr>
</thead>
<tbody>
<tr>
<td>V (volts)</td>
<td>7.58</td>
<td>7.56</td>
<td>7.55</td>
<td>7.53</td>
<td>7.51</td>
<td>7.47</td>
<td>7.44</td>
<td>7.28</td>
</tr>
</tbody>
</table>

Table 5.8: Measurement of flow velocity.

Fit the given data using the relation given below and calculate the unknown coefficients.

\[ u = A(e^{BV}) \]  

(A-56)

Q 18. The yield stress of many metals, \( \sigma_y \), varies with the size of the grains. Often, the relationship between the grain size, \( d \), and the yield stress is modelled with the Hall-Petch equation.

\[ \sigma_y = \sigma_0 + k d^{-1/2} \]  

(A-57)

<table>
<thead>
<tr>
<th>d (mm)</th>
<th>0.006</th>
<th>0.011</th>
<th>0.017</th>
<th>0.025</th>
<th>0.039</th>
<th>0.060</th>
<th>0.081</th>
<th>0.105</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_y ) (MPa)</td>
<td>334</td>
<td>276</td>
<td>249</td>
<td>235</td>
<td>216</td>
<td>197</td>
<td>194</td>
<td>182</td>
</tr>
</tbody>
</table>

Table 5.9: Measurement of flow velocity.

Determine the constants and best fit the data points.
Bibliography


Chapter 6

Simple Harmonic Motion is Observed through Webcam

Waqas Mahmood, Sohaib Shamim, Sabieh Anwar and Wasif Zia

Look at things around you. From galaxies and stars in the skies to the inner workings of an atom, we observe never-ending motion. In this experiment we are going to study the simplest form of periodic motion—namely simple harmonic motion (SHM) using a readily available webcam and Matlab.

KEYWORDS Simple Harmonic Motion, Amplitude, Acceleration, Angular Frequency, Damping

APPROXIMATE PERFORMANCE TIME 4 hours.

6.1 Conceptual Objectives

In this experiment, we will,

1. understand simple harmonic motion and its damping under various conditions;

2. start making simple measurements of lengths and understand the parallax error;

64
3. understand how errors propagate from an observed to an inferred quantity;

4. identify dimensions and units for simple physical quantities and transform between physical and logical coordinates;

5. fit experimentally observed curves with mathematically modelled solutions;

6. perform simple image processing and computational tasks on the personal computer; and last,

7. understand the formation and display of colour on TV and computer screens.

### 6.2 Experimental Objectives

The aim of the present experiment is to examine the amplitude of a freely oscillating as well as an underdamped harmonic oscillator. The experiment requires measuring the damping constant and then making a quantitative comparison of theory with experimental results. We will also keep a keen eye on errors as we go along.

### 6.3 Theoretical Introduction

Stand against a light source. Tie a tennis ball to a string and whirl it over your head in a horizontal circle. Observe the shadow on the wall that is opposite to the light source. You will see linear motion of the shadow which is slowest on the edges (where it turns around) and fastest in the center. This behavior, slowing down when moving away from the center and speeding up when approaching the center, is the signature of simple harmonic motion (SHM).

**Characteristic Equation of SHM**

Now let’s analyze the above example in somewhat mathematical detail. The motion of the shadow of the ball cast on the wall can be described by,

$$ x(t) = x_0 \cos(\omega t + \phi), \quad (A-1) $$

where \( x(t) \) is the position of the shadow on the wall, \( x_0 \), called the amplitude, is the maximum distance of the shadow from the center, \( \omega \), the rate at which you
Figure 6.1: Position, velocity and acceleration in SHM. (a) The position \( x \) of the shadow, (b) the slope of \( x \) gives the velocity \( v \) and (c) the slope of \( v \) gives the acceleration \( a \).

are rotating the ball, \( t \) is the time and \( \phi \), the phase, is the deviation of the wave from a reference. (Also see Fig. 6.1.)

The gradient (slope) of Equation (A-1) gives the velocity of the shadow and the sign shows the direction of the velocity. The negative means that as \( x \) increases, \( v \) decreases. So the shadow has maximum velocity at \( x = 0 \) and minimum velocity (in fact momentarily zero) at \( x = x_0 \).

\[
v(t) = \frac{dx(t)}{dt} = -\omega x_0 \sin(\omega t + \phi). \tag{A-2}
\]

This notion of changing velocity can also be expressed in terms of the acceleration, given by the gradient (slope) of Equation (A-2).

\[
a(t) = \frac{d^2x(t)}{dt^2} = \frac{dv(t)}{dt} = -\omega^2 x_0 \cos(\omega t + \phi). \tag{A-3}
\]

We can replace \( x_0 \cos(\omega t + \phi) \) in the above Equation with Equation (A-1) and as a result obtain,

\[
a(t) = \frac{d^2x(t)}{dt^2} = -\omega^2 x(t). \tag{A-4}
\]

This is the characteristic differential Equation describing SHM.

**Q 1.** Write down the Equation for the motion of the particle which is released from rest at 2 meters from origin, \( O \) in the positive direction and first returns to this position after 4 seconds. Also sketch the motion [1].

**Q 2.** A balloon performs SHM in a vertical line with a period of 40 seconds. Its height varies between 800 and 850 meters. Find the speed of the balloon when it is at 820 meters [1].
6.3. THEORETICAL INTRODUCTION

Q 3. A particle is moving with SHM. At what distance from the center will the velocity be half that of the maximum velocity?

Mass-Spring System

Combining Newton’s law,

\[ F = ma \]  \hspace{1cm} (A-5)

and Hooke’s law, \(^1\)

\[ F = -kx \]  \hspace{1cm} (A-6)

where \( k \) is the spring constant, we get

\[ a = -\frac{k}{m}x. \]  \hspace{1cm} (A-7)

This can also be written as,

\[ \frac{d^2x}{dt^2} + \omega_0^2 x = 0, \]  \hspace{1cm} (A-8)

with the definition,

\[ \omega_0^2 = \frac{k}{m}. \]  \hspace{1cm} (A-9)

Now let’s analyze the energy of this system. The total mechanical energy \( (E_T) \) of the mass-spring system is the sum of kinetic and potential energies, \( E_K \) and \( E_P \).

\[ E_T = \frac{1}{2}mv^2 + \frac{1}{2}kx^2. \]  \hspace{1cm} (A-10)

that can be re-written in light of Equation (A-2) as,

\[ E_T = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2. \]  \hspace{1cm} (A-11)

The conservation of energy ensures that the total energy is unchanging. As a result, kinetic and potential energies inter-convert keeping the sum constant. At the point of maximum displacement, the energy is wholly potential \( (E_K = 0, E_T = E_P) \) and at the centre, all the energy is kinetic \( (E_P = 0, E_T = E_K) \).

Q 4. What are the units of \( \omega, k \) and \( m \)?

\(^1\)“It is very evident that the Rule or Law of Nature in every springing body is, that the force or power thereof to restore it self to its natural position is always proportionate to the distance or space it is removed therefrom, whether it be by rarefaction, or separation of its parts the one from the other, or by a Condensation, or crowding of those parts nearer together.” Robert Hooke (1678)
Q 5. Sketch a graph between \( E_r \) and time. Now on the same graph sketch curves for \( E_k \) and \( E_p \).

Q 6. A spring of natural length 0.6 meters is attached to a fixed point \( A \) on a smooth horizontal table in the presence of air. A force of 10 \( x \) N is needed to keep the spring extended by \( x \) meters. A block of mass 0.1 kg is attached to the other end. The block is pulled away from \( A \) until it is 0.75 meters from \( A \), and then let go. Describe the subsequent motion [1].

**The Decay of Vibrations**

Equation (A-4) represents the ideal situation where there is no damping. A mass set in motion once will continue displaying its oscillations for all times to come. However, as we all very well know, this is never the case. The amplitude of the oscillating mass keeps on decreasing, till it eventually comes to rest. We now investigate this behaviour, called the *damping effect*.

Every medium (such as air, water) exerts a frictional forces on any moving object. This resistive force slows down the object, eventually bringing it to rest. In our case, the medium is air, slowing down the oscillating mass-spring system. One form of Newton’s second law that models this frictional effect is,

\[
F = -bv.
\]  

(A-12)

where \( v \) is the velocity of the particle and \( b \) is the *drag coefficient*.

Combining the two forces (-\( kx \) and -\( bv \)), we get,

\[
F = -bv - kx = ma.
\]  

(A-13)

\[
\Rightarrow \quad a + \frac{b}{m} v + \frac{k}{m} x = 0.
\]  

(A-14)

This can be re-written as,

\[
\frac{d^2 x(t)}{dt^2} + \frac{b}{m} \frac{dx(t)}{dt} + \frac{k}{m} x(t) = 0,
\]  

(A-15)

Replacing \( \frac{b}{m} \) and \( \frac{k}{m} \) with \( \gamma \) and \( \omega_0^2 \) respectively, the differential Equation now becomes

\[
\frac{d^2 x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = 0.
\]  

(A-16)

Q 7. What is the difference between Equations (A-8) and (A-16)?
Figure 6.2: Decaying oscillations for underdamped, overdamped and critically
damped harmonic oscillations. Our experiment is performed in the underdamped
regime.

In this particular case, the damping is characterized by $\gamma$ and $\omega_0$ represents the
natural angular frequency of the system if damping were absent.

In the current experiment, we will consider only one kind of damping: under-
damped SHM. As the term underdamped suggests, the oscillations die away,
albeit slowly. We will not go into the details of the various kinds of damping.
However Figure 6.2 gives a good intuitive feel of how these damped motions look
like.

The solution to Equation (A-16), for the underdamped oscillations, is given by
($\phi = 0$).

$$x(t) = A \exp\left(-\frac{\gamma}{2} t\right) \cos(\omega_1 t). \quad (A-17)$$

where $A$ is the maximum amplitude at $t = 0$ and,

$$\omega_1 = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}. \quad (A-18)$$

The solution, Equation (A-17) is the mathematical representation of the under-
damped oscillation of the mass-spring system, it shows how the position $x(t)$
vary as time $t$. This solution can be divided into two parts, $A \exp\left(-\frac{\gamma}{2} t\right)$ repre-
sents the damping of the amplitude (also called the envelope) whereas $\cos(\omega_1 t)$
represents the oscillations of the mass, with a frequency $\omega_1$. Note that $\omega_1 \neq \omega_0$. 
6.4 Apparatus

Our damped harmonic oscillator apparatus consists of a set of masses (locally fabricated) attached to the end of a helical spring (PASCO part no. SE-8749). The mass is displaced slightly and the resulting motion is recorded using a web camera (A4Tech) that is attached to a computer. The data processing is performed in Matlab that is equipped with the Image Processing toolbox. Fig. 6.3 shows the schematic setup of the experiment.

![Schematic diagram of the experimental setup.](image)

6.5 The Experiment

**Determining the Spring Constant**

★ Q 8. Note the mass \( m \) of the mass hanger. What are its units?

★ Q 9. Set up the apparatus as shown in the schematic. Attach the mass hanger and note the extension, using the attached meter rule. What are the units of the extension?

★ Q 10. Add weights of equal mass and measure the extension each time a
weight is added. You must not exceed the elastic limit.

★ Q 11. Plot a graph using the table developed above and draw the least squares curve fit.

★ Q 12. Find the spring constant \(k\). What are the units of \(k\)?

★ Q 13. What is the error in \(k\)?

★ Q 14. Calculate an expected value of \(\omega_0\) using Equation (A-16). Also predict the period of free oscillations. What is the uncertainty in the time period?

**Free Oscillations**

Remove the meter rule and start Matlab. (In the script, Matlab commands are preceded with the symbol \(\gg\).)

Pull the mass down by about 5 cm and release it.

\(\gg \text{webcam}\)

The command above activates a live video preview window and you can see the mass oscillating on your computer screen. Readjust the apparatus to obtain a vertically oscillating mass appearing on the computer screen.

In order to acquire frames, you need to type

\(\gg \text{start(vid);}\)

This will acquire 300 frames. Observe the parameters updating themselves in the bottom of the video preview window. In order to store these frames in the memory, you need to type

\(\gg \text{savedata}\)

This will create two arrays named frames and time and store the acquired frames and their time stamped values.

\(\gg \text{size(frames)}\)

★ Q 15. You see 4 numbers. Write down these values and explain what they represent.

The first frame can be viewed by typing in the following commands.
frame = frames(:, :, :, 1);

imview(frame);

Q 16. What is the size of the first frame?

Try viewing the second, third and the fourth frame.

Next you need to crop the acquired frames so that all the irrelevant information is filtered out. Matlab has a built-in command known as imcrop which crops an image to a specified rectangle. You can crop the first frame by typing

rect = \[x_{min}, y_{min}, (x_{max} - x_{min})/\text{width}, (y_{max} - y_{min})/\text{height}\]

HINT: Use the data cursor to scan the x and y coordinates for their maximum and minimum values.

cropregion = imcrop(firstframe, rect);

Q 17. How will you view the cropregion?

Q 18. Now write a Matlab code that crops all the 300 frames in one go and saves the cropped array with the name regions. HINT: Make a for loop.

Q 19. What is the size of the array regions?

imageprocess

will process all the cropped images in regions and will calculate the centre of mass of the oscillating mass. In order to calculate the displacement coordinates for the centre of mass, use the command

masscentre

The x and y coordinates of the centre of mass through the 300 frames can be viewed by typing

x = centre(:, 1), and

y = centre(:, 2).

Q 20. Plot a graph of the x coordinates against time. What does this graph physically represent?

Q 21. What happens to the x coordinates and explain why?
6.5. **THE EXPERIMENT**

- **Q 22.** Now plot a graph of the $y$ coordinates with time. Record the approximate time period of the oscillation in seconds.

- **Q 23.** Compare this result with the time period from Question 14.

- **Q 24.** When the undamped spring is stretched and released, show that the total energy $E_T$ at any given time is,

\[
E_T = \frac{1}{2} m x_0^2 \omega_0^2 = \frac{1}{2} k x_0^2.
\]  

(A-19)

HINT: Use $v = -x_0 \omega \sin(\omega t)$ and Equation (A-10).

**Damped Oscillations**

- **Q 25.** We now move on to see how the energy changes with time in an underdamped system with $\omega_0 \gg \gamma$. Show that the energy at any subsequent time is given by,

\[
E(t) \approx E_0 \exp(-\gamma t)
\]  

(A-20)

where,

\[
E_0 = \frac{1}{2} m A^2 \omega_0^2 = \frac{1}{2} k A^2.
\]  

(A-21)

- **Q 26.** Place the beaker containing water under the mass-spring system. Pull the mass down so that it is completely immersed in water. Repeat the experiment as in Section 6.5.

- **Q 27.** Plot the graph of amplitude against time. What do you observe?

- **Q 28.** From the graph, record the value of amplitude of each cycle and the corresponding time at which this happens.

- **Q 29.** Plot a graph of $\ln\left(\frac{E(t)}{E_0}\right)$ versus time, where $E_0$ represents the energy of the first oscillation that you took into consideration and $E(t)$ represents the energy at any given time thereafter.

- **Q 30.** Why do you think plotting a graph of $\ln\left(\frac{E(t)}{E_0}\right)$ against time is more appropriate than a graph of $\ln[E(t)]$ against time?

- **Q 31.** Using the available data, calculate the estimated values for $\gamma$, $\omega_1$ and hence for $b$. 
Q 32. Suggest ways in which the experiment can be improved.

6.6 Experience Questions

1. Is jumping on a spring mattress SHM? Explain your answer.

2. Are there any springs in nature?

3. Would a mass-spring system also work in space?

4. Can we make a spring using strong magnet bars?

5. Why do we need shock absorbers with springs in cars?

6.7 Idea Experiments


2. Solve the differential Equation (A-15) using the quadratic formula and analyze the discriminant for less than, equal to and greater than zero.

3. Try horizontal mass-spring system instead of gravity driven. Discuss friction.

4. Try to find out if friction is surface area dependent or not. HINT: Slide blocks down a plank and observe using a webcam.

5. Study the diffusion of ink in water using webcam [2].

6. Demonstrate the damping of a pendulum [3].
Bibliography


Chapter 7

Rotational Dynamics,
Moment of Inertia, Torque
and Rotational Friction

Waqas Mahmood, Sohaib Shamim, Wasif Zia and Sabieh Anwar

Ask a friend to help you rotate on a computer chair with your arms and legs stretched outwards. Keeping your body stiff, pull your legs up and wrap your arms around while you are still rotating. What do you notice? What was the feeling? How does it change when you pulled back your limbs?

Considering that there is negligible resistance by the chair pivot and air, push gently. There is some resistance that your partner’s mass poses when you try to rotate him/her. This feeling of opposition is the property of mass called inertia which resists change of state; rest or uniform motion.

Well developed Newtonian Mechanics is all that was applied by National Aeronautics and Space Administration (NASA) to reach the Moon. From our car engines and heavy industrial equipment to celestial bodies, all follow Newtonian Mechanics. Nevertheless we cannot regard Newton’s laws as universal because Relativistic Mechanics and Quantum Mechanics are more general and better at explaining nature in their own rights.
KEYWORDS

Rigid Body · Angular Momentum · Angular Velocity · Angular Acceleration · Moment of Inertia · Torque

7.1 Conceptual Objectives

In this experiment, we will,

1. appreciate the similarities and differences between rotational and translational motion;

2. investigate energy loss through friction;

3. process and analyze data to extract required information;

4. appreciate that there exist different ways of measuring a physical quantity with different accuracy and precision;

5. fit experimentally observed curves with mathematically modeled solutions; and

6. see how errors propagate from measured to inferred quantities.

7.2 Experimental Objectives

The experiment is divided into four sections. The early sections require making measurements of angular velocity and moment of inertia of the available disks, which are subject to frictional losses. Later sections require the use of a computer to help you monitor and record these measurements more quickly.

7.3 Theoretical Introduction

This experiment introduces you to the concepts of rotational motion. We shall touch upon a number of topics and discuss how a large complex object can be considered to be composed of a large assemblage of ideal particles. We will elaborate that a full description of a body’s motion must include linear as well as rotational motion. Furthermore, we will discuss torque as it applies to our experiment.
Angular Momentum

We can consider the provided circular disks (rigid bodies) to be made up of small infinitesimal particles of masses \(m_1, m_2, m_3, \ldots m_i \ldots\). Their placement may be defined with the position vectors \(r_1, r_2, r_3, \ldots r_i, \ldots\) and when rotating, their instantaneous velocities may be defined as \(v_1, v_2, v_3, \ldots v_i, \ldots\). The index \(i\) shows one of the many particles.

Figure 7.1 illustrates the \(i\)th particle rotating about the \(z\) axis.

![Angular Momentum Diagram](image)

Figure 7.1: A representative particle rotating about the \(z\)-axis; \(m_i v_i\) is the linear momentum and \(J_i\) is the angular momentum.

The angular momentum of the particle about \(z\) axis is given by,

\[
J_i = m_i v_i \times r_i, \quad (A-1)
\]

where \(\times\) denotes the vector or cross product.

For a particle rotating with an angular velocity \(\omega\) about \(z\) axis, we can say that,

\[
v_i = r_i \omega, \quad (A-2)
\]

Consider a circular disk rotating about the \(z\) axis. The disk itself can be considered to be composed of with all its infinitesimal elements in the \(xy\) plane. Using Equation (A-1) and Equation (A-2) we can write for the \(i\)th particle,

\[
J_i = m_i r_i^2 \omega. \quad (A-3)
\]

The total angular momentum of a disk about an axis is simply the sum of all the angular momentums for the infinitesimal particles.
Figure 7.2: Disk can be considered to comprise particles. The individual angular momentums of these particles will all add up resulting in the total angular momentum.

\[ J = \sum_{i=1}^{N} m v_i \times r_i. \]  \hspace{1cm} (A-4)

**Moment of Inertia**

The cross product for a disk rotating about the \( z \) axis with its components in the \( xy \) plane can be expanded as,

\[ J_z = \sum_{i=1}^{N} (m r_i^2) \omega. \]  \hspace{1cm} (A-5)

Here \( I = \sum m r_i^2 \) is a constant (irrespective of the angular velocity of the disk) and is known as the moment of inertia of the disk. Therefore Equation (A-5) becomes,

\[ J_z = I \omega. \]  \hspace{1cm} (A-6)

In the present experiment, we will investigate the rotational kinematics of a disk. It will also be helpful to know the moment of inertia for a circular disk, which is,

\[ I = \frac{1}{2} M R^2 \]  \hspace{1cm} (A-7)

where \( M \) is the mass and \( R \) is the radius of the disk.

**Q1.** Show that the kinetic energy of a rotating disk is given by,

\[ KE = \frac{1}{2} I \omega^2. \]  \hspace{1cm} (A-8)
(HINT: Kinetic energy for a particle is given by, \( K = \frac{1}{2}mv^2 \). Sum for all the particles and use the fact, \( I = \sum m r^2 \).)

Moment of inertia is analogous to \textit{inertia} in linear kinematics. However, since in rotational motion, we always find ourselves dealing with moments, we call the inertia in circular motion as \textit{moment of inertia}. Moment of inertia of a particular body is defined with respect to a particular rotation axis and is different for a body when it is rotating about \( x \), \( y \) or \( z \) axes. Table 1 provides a brief comparison of linear and rotational motions and their characteristics.

<table>
<thead>
<tr>
<th>Concepts and quantities</th>
<th>Linear Motion</th>
<th>Rotational Motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Position</td>
<td>( x )</td>
<td>( \theta )</td>
</tr>
<tr>
<td>Velocity</td>
<td>( v )</td>
<td>( \omega )</td>
</tr>
<tr>
<td>Acceleration</td>
<td>( a = \frac{v}{t} )</td>
<td>( \alpha = \frac{\omega}{t} )</td>
</tr>
<tr>
<td>Motion Equations</td>
<td>( x = vt )</td>
<td>( \theta = \omega t )</td>
</tr>
<tr>
<td>Newton's 2\textsuperscript{nd} Law</td>
<td>( F = ma )</td>
<td>( \tau = I\alpha )</td>
</tr>
<tr>
<td>Momentum</td>
<td>( p = mv )</td>
<td>( J = I\omega )</td>
</tr>
<tr>
<td>Work</td>
<td>( Fx )</td>
<td>( \tau\theta )</td>
</tr>
<tr>
<td>Kinetic Energy</td>
<td>( \frac{1}{2}mv^2 )</td>
<td>( \frac{1}{2}I\omega^2 )</td>
</tr>
</tbody>
</table>

\textbf{Q 2.} A circular disk of mass 0.2 kg and radius 0.1 m is rotating at 10 revolutions per second. Calculate the angular frequency, moment of inertia and kinetic energy of this disk.

\textbf{Turning Effect}

\textbf{Q 3.} Define torque.

Mathematically, torque (\( \Gamma \)) is,

\[
\Gamma = r \times F, \quad (A-9)
\]

where \( r \) is the displacement between the line of action of force and the particle and \( F \) is the force applied.

Expanding the cross-product we get,

\[
\Gamma = rF \sin(\theta), \quad (A-10)
\]

where \( \theta \) is the angle between \( F \) and \( r \).
In a gravity driven system, we may replace $F$ using Newton’s second law and express the equation as,

$$
\Gamma = mg r \sin(\theta),
$$

(A-11)

where $m$ is the mass used to drive the mechanism and $g$ is the acceleration due to gravity.

### Angular Acceleration

**Q 4.** Define angular acceleration.

Mathematically, angular acceleration $\alpha$ is given by

$$
\alpha = \frac{d\omega}{dt},
$$

(A-12)

You may want to refer to Table 7.1 to become more comfortable with this seemingly new term which is just an equivalent of linear acceleration adapted for rotational motion. In other words it is the gradient of angular velocity versus time graph.

Newton’s second law for rotational motion states that,

$$
\Gamma = l \alpha
$$

(A-13)

where $\Gamma$ is the applied torque, $\alpha$ is the angular acceleration and $l$ is the moment of inertia—the rotational equivalent of mass. Note its similarity to Newton’s law for linear motion $F = ma$, establishing the moment of inertia as the analogue of mass and torque as the analogue of force.

If we substitute Equation (A-11) and Equation (A-12) in Equation (A-13) we get,

$$
mgr\sin(\theta) = l \frac{d\omega}{dt}.
$$

(A-14)
7.4 Introduction to the Apparatus

In addition to the PC, the apparatus consists of the following components:

1. **Rotational Motion Apparatus** This apparatus, ME-9341 was procured from PASCO Scientific. The contents of the apparatus are in Figure 7.3.

![Diagram of apparatus components](image)

Figure 7.3: The components of the rotational motion apparatus. Note the arrows showing the particulars of the components.

2. **Hand–Heled Interface and Photogate** An interface from Vernier LabPro has been used to transfer our data to LabVIEW as the apparatus was not originally designed to run on a PC. An additional photogate which is calibrated for the interface is also used in the later section.

The photogate sends a narrow beam of infrared radiation from one arm which is detected by a detector in the opposite arm. When the beam is
first blocked, a signal is sent to the smart timer \((ST)\). The Gate mode records the time between the two successive blocking of infrared light and helps in measuring the speed and average velocities of an object passing through the photogate as shown in Figure 7.4.

We use Fence mode to record the time between ten successive interruptions of the photogate. The timing begins when the beam is first blocked and stops when it has been blocked ten times. Using the Select Measurement key, user can recall the 10 different times when the beam was blocked.

![Diagram of Gate Mode](image)

**Gate Mode**

![Diagram of Fence Mode](image)

**Fence Mode**

*Figure 7.4: Illustration of the modes for the Smart Timer.*
7.5 Experimental Method

Preparation

★ Q 5. Place the bubble level on the base. See if the bubble is in the inner ring. If not, screw or unscrew the three adjustable supports underneath, to bring the bubble in the inner most ring.

★ Q 6. Find the diameters of all the pulleys on the main platter.

★ Q 7. Find the mass of the main platter. Note the uncertainty in your reading.

★ Q 8. Slide the spindle into the bushing of the bearing assembly. Then slide the main platter atop with the pulleys facing up. Then attach a cardboard or chart paper strip to it as shown in the Figure 7.6.

Figure 7.6: Cardboard attached to the disk.

★ Q 9. Clamp the super pulley (SP) to the base such that the thread from the step pulley is in line with the edge of the SP.

★ Q 10. Attach the screw rod to the photogate (P₁) and slide it into one of the holes on the sides of the base.

★ Q 11. Check the connections of the photogate (P₁) with the smart timer (ST).
7.5. EXPERIMENTAL METHOD

★ Q12. Check if ST beeps when you switch it ON using the power switch on the side. Press 1 and then 2. You will now be in the GATE timing mode.

★ Q13. Give a push to the main platter and press 3 from ST.

★ Q14. Explain the value that has been returned by ST.

★ Q15. What is the uncertainty in your measurement?

Determination of Moment of Inertia /

★ Q16. Take a thread of suitable length e.g. 150 cm approximately, the mass hanger weighing 5 g and a 100 g mass. Tie one end of the thread to the hanger and wrap the other end around the medium or middle pulley. Now pass this thread over the SP.

★ Q17. Change the measurement mode of the ST to FENCE. This can be achieved by pressing 1 once and then 2 twice from ST. In the FENCE mode, the ST is capable of taking ten readings in one go.

★ Q18. Perform the following experimental procedures.

1. With help of the provided meter rule, note the height from which you want to release the mass.

2. Set the system in motion.

3. Wait for the hanger to hit the ground.

4. Immediately after this, press START from ST which measures the time period for ten rotations.

★ Q19. Switch off the ST.

★ Q20. Repeat the procedure with the stop watch and record the time period for first, sixth and tenth rotation.

★ Q21. Tabulate your results and calculate the angular speed of the disk using the ST and stopwatch data. Explicitly write down the uncertainties in your results.

The rotational kinetic energy gained by the disk equals the gravitational potential energy lost by the masses.
Q 22. Express the above statement in mathematical form, i.e. in the form of an equation.

Q 23. Determine the moment of inertia of the main platter.

Q 24. What is the uncertainty in the value of I?

Q 25. What is the main source of error in your measurement?

Show your results to the demonstrator once you have reached this point.

Frictional Losses

Q 26. Take a thread of suitable length e.g. 100 cm, a mass hanger and a one or two 150 g masses. Tie one end of the thread to the screw protruding from the smallest pulley on the main platter. Then pass the thread from the holes right beside the screw. Wind the rest of the thread on the medium or middle pulley and tie the other end to the hanger.

Q 27. Perform the following procedural steps.

1. Measure the diameter of the main platter.

2. Attach the cardboard with 32 equal spacings onto the main platter as shown in the Figure 7.7.

![Diagram of cardboard paper with photogate, main platter, 4mm thick strip, smart pulley, and mass + hanger (150 g)](image)

Figure 7.7: Calculating frictional losses using the cardboard method.

3. Set the photogate \( P_1 \) in counts (Manual)\(^1\) mode. This ensures that ev-

\(^1\) In manual mode, counter tickles on every count and user can stop it manually at any time.
7.5. **EXPERIMENTAL METHOD**

Every time the photogate beam is blocked by the cardboard arm, a count is recorded. You can then use the separation between the strips and the number of counts to work out the vertical distance moved by the mass.

4. Allow the mass to fall down and note the number of counts at maxima and minima until the oscillations die out.

5. Use the number of counts to work out the height lost during each oscillation.

6. Repeat the experiment **at least** three times and determine the average energy lost in one revolution.

![Figure 7.8: Maximum and minimum heights.](image)

**Q 28.** Now refer to your results in the previous section and determine a new value for the moment inertia of the disk.
Decay of Angular Velocity

For the remainder of this experiment, we will use a PC to record data. Ask the demonstrator to set-up the photogate \( P_2 \) for you.

**Q 29.** Load the data acquisition program by double clicking on the LabVIEW file `rotationaldynamics`.

**Q 30.** You need to enter a file path where your data will be stored. Enter the path `Z:\Your Preferred Folder`. You can start the program by clicking the **RUN** button.

**Q 31.** LabVIEW will ask you for a file name where it will save data. Enter `velocitydecay` as the file name.

**Q 32.** Attach the provided card to the main platter. Now using the provided thread, rotate the platter with the maximum possible angular velocity. (You will need to pull the thread quickly and forcefully!)

**Q 33.** Press the **Acquire** button. You can stop the acquisition at any time by pressing the **Stop** button.

**Q 34.** You can see the time elapsed since the start of the acquisition under the heading **Indicators**. The green light flashes every time LabVIEW acquires data. If you want to see the acquired data on the LabVIEW screen, press the **Plot Graph** button.

**Q 35.** Stop the acquisition when the platter has almost come to a halt.

**Q 36.** Open Matlab. Import all your data into Matlab by typing

\[
>> \text{rotationalmechanics}
\]

in the Matlab command prompt. You can see two columns, one showing the number of times the strip on the main platter has crossed the photogate and the other showing the corresponding time.

**Q 37.** Analyze your data to obtain angular velocities at different times and plot a graph of angular velocity against time. Think carefully if you should use all of the data when plotting a graph.

The decay in angular velocity may be modeled by the exponential function:

\[
\omega = \omega_0 \exp(-ct)
\]

(A-15)
where $c$ is a positive constant.

★ Q 38. Plot least squares curve fit to the date points following the Equation (A-15).

★ Q 39. Plot the natural log of $\omega$ with respect to time and through curve fitting and estimate the value of $c$.

### 7.6 Experience Questions

1. Why do we feel vertigo after a spin on a merry-go-round?

2. Does the inertia of our body increase in a swimming pool?

3. If mass $m$ is right on top of center of gravity, does that mean there is no value of $I$ for that mass?

4. In an accident, the body may be stopped by the seat belt and air bags. Where does all the momentum go? Can this cause bodily injury?

5. Why do cats always fall feet first on the ground?

### 7.7 Idea Experiments

1. Find your body’s moment of inertia.

2. Use different masses to check if the relationship between the frictional losses and energy delivered is linear.

3. Drop the auxiliary platter on the main platter when rotating to verify the conservation of momentum.
Chapter 8

Heat Transfer and Newtons Law of Cooling

Sabileh Anwar, Sohaib Shamim and Wasif Zia

If you put one end of a spoon on the stove and wait for a while, your finger tips start feeling the burn. So how do you explain this simple observation in terms of physics?

We all know that flowing matter (such as air) in contact with a heated object can help ‘carry the heat away’. The motion of the fluid, its turbulence, the flow pattern and the shape, size and surface of the object can have a pronounced effect on how heat is transferred. These heat flow mechanisms are also an essential part of our ventilation and air conditioning mechanisms, adding comfort to our lives. Importantly, without heat exchange in power plants it is impossible to think of any power generation, without heat transfer the internal combustion engine could not drive our automobiles and without it, we would not be able to use our computers for long time and do lengthy experiments (e.g Heat Transfer), without overheating and frying our electronics. Heat transfer is also an integral component of the global climatic cycle, affecting how the human civilization has demographically placed itself on the globe and what lifestyles and customs have evolved around geographical habitats. Finally, global warming is a slow poison that will, in part, determine our future destinies.

KEYWORDS
8.1 Conceptual Objectives

In this experiment, we will,

1. Learn about Newton's law of cooling, and simultaneous radiative and convection losses;
2. identify the role of thermally conducting and insulating materials;
3. learn about temperature measurements using thermocouples;
4. corroborate experimental results with theoretical predictions;
5. mathematically model natural processes;
6. appreciate the role of approximations in experimental science; and
7. calculate the propagation of errors from observed to inferred quantities.

8.2 Experimental Objectives

In the present experiment, we heat an object and observe how it cools with time and what factors affect the cooling rate. We adapt the experimental setup to interchange between two different environments. In one section, we allow the object to be cooled with the help of forced air currents and in the other, the system is made to act like a black body cavity. We will also learn how to use the thermocouple, an important component of numerous commercially important processes.

8.3 Theoretical Introduction

Thermal Convection

Suppose you are driving your car in a hot June afternoon. You bend over a bit to see the air above your car's hood. Why does the background seem so hazy? The
Observation is a result of a process called convection and it occurs when a moving fluid comes in contact with an object whose temperature is higher than that of the fluid itself. When the less energetic molecules of the air come in contact with the fast vibrating molecules of the hood, they undergo collisions, picking up energy from the hot surface of the hood. At the intimate interface of the hood and the air, the process is exactly similar to conduction. But the temperature of the air soon rises at the surface, the density decreases and the molecules have become more buoyant, causing the hot air to rise. These molecules then transfer the thermal energy to neighboring molecules through collisions (conduction) as well as through the bulk flow of air (convection). In practice, both of these modes of heat transfer go on, hand in hand [1]. Which process dominates is determined by the shape of the heated object and the flow velocity and profile of the fluid.

Convection is also seen at the global scale when it rains. In fact in Lahore, we all eagerly await the Monsoon season. It is the process of convection that transports the thermal energy from the hot land surfaces to the atmosphere. The rising hot air on the land creates a low pressure region that sucks air laden with condensed water vapour from above the Bay of Bengal and the Arabian Sea. By the time clouds reach the land mass, they gradually rise to higher and higher altitudes, the moisture is condensed and the clouds finally lay their watery burden onto the thirsty land.

**Newton’s Law of Cooling**

Newton’s law of cooling states that the rate of change of temperature of an object is directly proportional to the temperature difference of the object with its surroundings. Mathematically,

\[ Q_{conv} = hA(T_2 - T_1). \]  \hspace{1cm} (A-1)

and in terms of power density,

\[ q_{conv} = h(T_2 - T_1). \]  \hspace{1cm} (A-2)

Here \( T_2 \) is the temperature of the hot object and \( T_1 \) is the temperature of the fluid far away from the object. The units of \( Q_{conv} \) are watts and \( h \) is called the coefficient of convective heat transfer. Equation (A-1) is sometimes referred to as Newton's law of cooling.

**Q 1.** What are the units of \( h \)?
8.3. THEORETICAL INTRODUCTION

A = \pi dL

\[ Q \]

\[ A = \pi dL \]

\[ d \]

\[ T_2 \]

\[ T_2 \]

\[ L \]

\[ L \]

\[ Q \]

\[ T_1 \]

\[ T_1 \]

Figure 8.1: Setting for Newton’s law of cooling. The power transmitted from a rod of surface area \( A \) is \( Q \). The surface is at a steady temperature of \( T_2 \) and \( T_1 \) is the temperature of a mass of air far away.

Amongst several other tasks, this experiment will help us determine (a) how \( T_2 \) varies with time, and (b) the value of \( h \). The value of \( h \) depends on the properties and flow of the fluid, the temperature of the hot surface, the surface geometry as well as the bulk fluid velocity [1]. It is an empirical quantity.

Q 2. Hot air at 80°C is blown over a 2 x 4 m² flat surface at 30°. If the average coefficient of convective heat transfer is 55 W m⁻², determine the rate of heat transfer from the air to the plate [1].

**Forced Convection**

Many electronic devices these days, computers included, come with cooling units. These are small fans that direct a stream of air onto the printed circuit board that is likely to get heated or the microprocessor. The increased air currents help the convection process, supplementing the density-assisted buoyant forces. Mathematically, forced convection, as it is called, changes the value of \( h \). For example, for convection in still air, the value of \( h \) could be \( 2 - 25 \) W m⁻² K⁻¹ whereas this could go as high up to 250 if the air is in motion.

Figure 8.2: Forced convection.
Interestingly, human bodies also produce heat. Ventilation systems in buildings are designed keeping in account the heat loads of human bodies. An average adult, even in a state of resting, has a certain basal metabolic rate (BMR). The process generates heat. The typical heat load is 90 W per person and this heat must be dissipated. For an average human surface area of 2 m², the flux of heat that must be transferred to the atmosphere is 45 W m⁻². We all know very well, that in summers, when it is very hot, it becomes increasingly difficult to dissipate this heat and hence most of us resort to the luxuries of forced convection. We must also remember that the human body has, in fact, developed a very sophisticated regulatory mechanism for this purpose.

Q 3. Air impinges onto a power transistor with a certain velocity, always maintaining a convective heat transfer coefficient \( h \) of 100 W m⁻² K⁻¹. The temperature of the air is 25°C and the maximum temperature the transistor can withstand is 60°C. The diameter and length are 10 mm each. Calculate the maximum power dissipation of the transistor? (Adapted from [2].)

![Figure 8.3: Power transistor.](image)

Q 4. You extend your hand outside a car moving at a speed of 60 km h⁻¹. The outside air temperature is 5°C and the air velocity results in a value of \( h \approx 50 \) W m⁻² K⁻¹. The skin temperature is 34°C, slightly lower than the normal internal body temperature. What is the maximum heat transfer rate this kind of forced convection can support? (Adapted from [2].)

Radiation (Stefan-Boltzmann Law and Cavity Radiation)

There is yet another mode of heat transfer. This mode does not require the presence of any medium or molecular interactions and is called radiation.

Every object in nature radiates and absorbs electromagnetic waves, be it day or night. How does light and heat, from the Sun, reach us? Even when there is no (real) matter in the space in between. Why is it that even with a cool breeze on the Clifton sea-front, the warmth of a bonfire keeps us cosy? The answer lies in radiation.
Radiation is a result of temperature. If a body is hotter than its surroundings it emits more radiation than it absorbs, and tends to cool; if a body is cooler than its surroundings it absorbs more radiation than it emits, and tends to warm. It will eventually come to thermal equilibrium with its surroundings, a condition in which its rates of absorption and emission of radiation become equal.

Suppose a solid object has a surface temperature $T_2$. The heat radiated per unit time is now denoted by $Q_{rad}$ and is given by,

$$Q_{rad} = \sigma A T_2^4.$$  \hspace{1cm} (A-3)

or in terms of the heat radiated per unit area per unit time,

$$q_{rad} = \sigma T_2^4.$$  \hspace{1cm} (A-4)

where $\sigma$ is a constant with a value of $5.67 \times 10^{-8}$ W m$^{-2}$K$^{-4}$, analogous to the $h$ we have discussed in the context of convection. This equation is generally referred to as the Stefan-Boltzmann law and an object respecting this condition is called a blackbody. A blackbody is a perfect emitter. Given a fixed temperature, no other object can emit more energy than a blackbody.

However, in practice, no real object is a perfect blackbody and the radiative power density $q_{rad}$ is decreased by a factor $\varepsilon$, called the emissivity. Equation (A-6) is modified to,

$$q_{rad} = \varepsilon \sigma T_2^4.$$  \hspace{1cm} (A-5)

An ideal value of $\varepsilon = 1$ refers to an object that emits all of the available radiative energy.

Now suppose, we place another very large surface (call it $Q$) that completely encloses the object of interest (call it $P$), as shown in Figure 8.4. The surface $Q$ emits at a lower temperature $T_1$, the output power density being,

$$q_{rad} = \varepsilon \sigma T_1^4.$$  \hspace{1cm} (A-6)

The net power density being transferred from $P$ to $Q$ becomes,

$$q_{rad} = \varepsilon \sigma T_2^4 - \varepsilon \sigma T_1^4 = \varepsilon \sigma (T_2^4 - T_1^4).$$  \hspace{1cm} (A-7)

and the radiative power transmitted is,

$$Q_{rad} = \varepsilon \sigma A(T_2^4 - T_1^4).$$  \hspace{1cm} (A-8)

Note that this power does not depend on the surface area of $Q$. 
Figure 8.4: A blackbody $P$ is placed inside another blackbody $Q$. The long arrows emanating outwards from $P$ represent the thermal power emitted by $P$ and the short arrows pointing inwards represent the thermal power absorbed by $P$.

**Q 5.** Is every black surface a blackbody?

**Q 6.** What assumptions go into writing Equation (A-7)?

**Q 7.** Do you expect a silvered mirror to have a high or low value of emmissivity?

Now suppose an object $P$ with emissivity $\varepsilon$ and surface area $A$ is heated to $T_2$ and placed inside the cavity. The temperature of the walls of the cavity and the cavity radiation is $T_1$ and $T_2 > T_1$. Both convection and radiation mechanisms are operative. The total heat energy lost by $P$ in unit time is given by the sum of the convective and radiative losses,

$$Q = Ah(T_2 - T_1) + \varepsilon \sigma A(T_2^4 - T_1^4). \quad (A-9)$$

As the object $P$ cools inside the cavity, its temperature $T_2$ reduces. If $c$ is the specific heat capacity and $m$ is the mass, the total heat lost by $P$ will be,

$$mc(T_{2,\text{initial}} - T_{2,\text{final}}) \quad (A-10)$$

and the rate at which the heat lost can be written as,

$$Q = -mc \frac{dT_2}{dt}. \quad (A-11)$$

where the minus sign shows heat being lost as temperature decreases. Comparing this with Equation (A-9), we obtain,

$$-mc \frac{dT_2}{dt} = Ah(T_2 - T_1) + \varepsilon A(T_2^4 - T_1^4). \quad (A-12)$$
and solving for the coefficient of convective heat loss,

\[ h = -\left( \frac{mc}{A} \right) \frac{dT}{dt} \approx \frac{\varepsilon \sigma (T_2^4 - T_1^4)}{T_2 - T_1}. \]  

(A-13)

This is a very important equation. Make sure you fully understand it and have re-worked the derivation. As \( P \) cools, the temperatures \( T_1 \) and \( T_2 \) both change with time. Therefore, we can also write these temperatures as \( T_1(t) \) and \( T_2(t) \).

8.4 Apparatus

Our apparatus is an enhancement over the experimental setup described in [2].

1. **Heating mechanism** We have adopted two heating methods for the experiment. The first is a locally fabricated furnace (*Adeel Electronics, Beden Road, Lahore*) which is set at 220°C. It is fitted with a heating element and a probe-type thermocouple that automatically cuts off the electric supply when the temperature goes above the specified value.

   You should be very careful and check for any current leakage using a tester before you touch it. Use thermal insulation gloves and the large tongs to transfer the cylinder into or out of the furnace.

   The second method is the hot plate. The object to be heated is placed inside a bath of graphite powder on a hot plate. The hot plate reaches a maximum temperature of 400°C. Both heating options are depicted in Figure 8.5.

![Heating mechanisms for the experiment: (a) furnace and (b) hot plate.](image)

Figure 8.5: Heating mechanisms for the experiment: (a) furnace and (b) hot plate.

2. **Cavity, Fan and Cylinder** The cavity for our experiment has been fabricated locally (Noor Trading and Contracting Co., Rawalpindi) and adapted in-
house. Both the walls of the cavity and the heated object (referred to hereafter as the cylinder) are made of mild steel oxidized at 800°C. The cavity has two inlets and is coated with a dull black paint inside for good radiative exchange (high value of emissivity \( \varepsilon \approx 0.8-0.9 \)). Beneath the cavity, we have fitted an exhaust fan 12 V DC, 0.93 A (Pak Fans).

The cylinder (5 in. in diameter) can be placed in a mount with a cushion of alumina silicate, a good thermal insulator to minimize heat loss by conduction.

3. **Lids: Perforated and Non-perforated** We have used two kinds of lids in the experiment. The perforated lid is used in the first half of the experiment where our primary mode of heat loss is forced convection, with the fan switched on. The non-perforated lid is used in the second half of the experiment; it reduces the convective currents so that our primary source of heat loss becomes radiation.

4. **Thermocouples** The experiment employs two thermocouples (Fanell). One thermocouple is attached to a clamp that can tightly grip the heated cylinder. The second thermocouple is suspended in air, near the walls of the cavity.

![Thermocouples](image)

Figure 8.6: Thermocouples used in the experiment.

5. **Data Acquisition System** The experiment uses standard data acquisition hardware. The data acquisition (DAQ) card (National Instruments PCI-6221) acquires, digitizes and amplifies the thermocouple voltage signal. These signals are routed through the signal conditioning unit (National Instruments SCC-68). The unit also houses a thermistor for hardware-based cold junction compensation.
8.5 Experimental Method

Newton's Law of Cooling

The schematic of the experimental setup is shown in Figure 15.2.

🌟 Q 8. In the presence of forced convection, it is generally believed that the radiative losses are negligibly small as compared to convective losses. With this assumption, the radiative terms can be dropped out from (A-13) and we recover Newton’s Law of cooling,

\[ h = -\left( \frac{mc}{A} \right) \frac{dT}{dT}. \quad (A-14) \]

Furthermore, if we assume that \( T_1 = \text{constant} \) (as done conventionally), we can make the substitution,

\[ T_2 - T_1 = x. \quad (A-15) \]

and after some algebraic manipulation, the equation becomes,

\[ \frac{dx}{x} = -\frac{hA}{mc} \, dt. \quad (A-16) \]

The solution is,

\[ x(t) = x_0 \exp\left(-\frac{hA t}{mc}\right). \quad (A-17) \]

where \( x_0 \) is the initial value of \( T_2 - T_1 \).
Assume $T_1$ is constant at its average value, $\langle T_1 \rangle$.

**Familiarization with the software**

- **Q 9.** Open the Labview VI `thermal.vi` by double clicking the shortcut located on the Desktop. A front panel window with a grey background will open.

- **Q 10.** Enter folder names where your data will be stored, for example

  ```
  C:\Documents and Settings\wasif.zia\Desktop\thermal
  ```

- **Q 11.** Enter the filenames for your data. You will make two files, one for the cylinder e.m.f $E_2$ and one for the cavity e.m.f $E_1$. Your files should be named `cylinder1` and `cavity1`. In this script, these files are also called the "data files".

- **Q 12.** Run the Labview file and use the data acquisition system to read the e.m.f values from the two thermocouples placed at room temperature. Observe the table of e.m.f readings as it is being populated. Ask your demonstrator for help if something is not clear.

**Experimental Procedure**

- **Q 13.** Measure the mass and surface area of the provided, black-coated and roughened mild steel cylinder. Note down your uncertainties.

- **Q 14.** With the demonstrator’s help, place the cylinder in the furnace (or on the hot plate) and heat it to about 350°C.

- **Q 15.** Now use the provided tongs and thermal gloves to **carefully** transfer the heated cylinder into the cavity. **Never touch the surface of the cylinder, or the hot plate with your bare hands. These are extremely hot surfaces.**

- **Q 16.** Now quickly follow the following steps, in the same order.

  1. Clamp the thermocouple marked ($E_2$) onto the cylinder.

  2. Switch on the fan.

  3. Run the VI by clicking the START button or by pressing CTRL+R.

- **Q 17.** Carefully observe the readings being picked by the programme.

- **Q 18.** Monitor the e.m.f generated till you get asymptotic values called $E_{eq}$, on both the thermocouples.
8.5. EXPERIMENTAL METHOD

Calibrating the thermocouples

The Labview programme returns e.m.f.'s generated by the thermocouples. These voltages determine the temperature. We need a conversion between the e.m.f. and the temperature. This is called calibration.

We can perform the calibration while the cylinder is cooling down. Heat about 1500 ml of water in a beaker to 80°C and then allow it to cool. Place a thermocouple connected to the GWInstek Digital Multimeter, and labeled, G, inside the beaker. Switch the knob on the multimeter so as to display the temperature in ºC. Place a different thermocouple, labeled E₃ in the beaker. Care should be taken so that E₃ and G are as close to each other as possible. Record the e.m.f generated by E₃ (the software also shows readings for E₃) and the temperature shown by G and establish a relationship between temperature and the generated e.m.f. You may want to use Matlab to process this data.

Use this calibration for the remainder of the experiment.

★ Q 19. Continue recording the data until the e.m.f shown by E₂ and E₁ are same. What does this condition represent? Has the transfer of heat ceased altogether? You can now stop the Labview programme, switch off the fan and focus on your data.

Data Analysis for Forced Convection

★ Q 20. Run MATLAB and change the path to the folder that contains your data files acquired from Labview.

★ Q 21. In the command window, type the command¹,

```matlab
>> forcedconvection
```

The m-file processes and filters your thermocouple measurements and generates the following vectors.

<table>
<thead>
<tr>
<th>E.m.f generated by T₂</th>
<th>E₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>E.m.f generated by T₁</td>
<td>E₁</td>
</tr>
</tbody>
</table>

The time values for the temperature data, t time

---

¹The file thermal.m should be in the same folder as the acquired data files.
Convert the e.m.f values to temperature by using the calibration performed in Q 18. From the data acquired, plot $T_2 - \langle T_1 \rangle$ versus time, where $T_2$ is the temperature of the cylinder and $\langle T_1 \rangle$ is the average room temperature, and fit the plot to the exponential function (A-17). Why are we justified in using (A-17)?

What is the value of $h$ based on the exponential fit? What is the error in your value of $h$?

Simultaneous Radiative and Convective Losses

In this last part of the experiment, we will seal the cavity, closing its base and covering it with the non-perforated lid at the top. The arrangement is depicted in Figure 15.3.

Slide the bottom cover gently down to the base of the cavity. Heat the cylinder again, carefully place it inside the cavity and put the non-perforated lid on top so that convection is reduced. Your files in the VI should be labeled cavity2 and cylinder2. Run the VI thermal_vii and leave it running till you reach asymptotic values.

Figure 8.8: Schematic sketch of the experimental setup for simultaneous convective and radiative heat losses.

Use the Matlab file thermal.m to generate the processed data. You will see numbers represented in 3 columns. The first column represents $\frac{dT_2}{dt}$, the second represents $T_1$ and the third $T_2$. Use these and Equation A-13 to compute the mean value of $h$. Plot a graph of $h$ versus temperature. Use $\varepsilon = 0.85$ and $c = 620 \text{ J kg}^{-1} \text{ K}^{-1}$. 

8.6 Experience Questions

1. Why is a light bulb hotter than tube light?

2. Are they really photons that warm us when we put our hands close to a heater?

3. Can we build a thermometer that measures temperature using colour?

4. Are there any animals with thermal vision?

5. Can we make fire by concentrating sun rays with a lens made of ice?

8.7 Idea Experiments

1. As water is heated, the temperature does not rise linearly. Design an experiment to measure the rise in temperature with time and describe your results in terms of Newton's cooling [4].

2. Measure the specific heat capacity of water through its cooling curve [5].

3. Is a white surface really a poor emitter of radiation? Compare the cooling curves of (a) an unpainted shiny metal, (b) a metal painted pitch black, and (c) a metal painted white.

4. Find out about the wall construction of the cabins of large commercial airplanes, the range of ambient conditions under which they operate, typical heat transfer coefficients on the inner and outer surfaces of the wall, and the heat generations inside. Determine the size of the heating and air-conditioning system that will be able to maintain the cabin at 20° at all times for an airplane capable of carrying 400 people [1].
Bibliography


Chapter 9

Magnetic Phase Transitions
of a Ferromagnetic Alloy

Wasif Zia and Sabieh Anwar

Magnetic phenomena have been known since time antiquity. The ancient Greeks knew about the magnetic force of both magnetite and rubbed amber. Magnetite, a magnetic oxide of iron mentioned in Greek texts as early as 800 B.C.E., was mined in the province of Magnesia in Thessaly. Thales of Miletus is considered to have been the first man to study magnetic forces. According to Lucretius, a Roman philosopher in the first century BC, the term magnet was derived from the province of Magnesia.

Starting from these early discoveries in magnetism, through the Chinese invention of the magnetic compass to the pioneering work of scientists such as Oersted, Ampere, Faraday, Maxwell and Néel, we have indeed come a long way in our understanding of magnetism. Today, a complete understanding of magnetism requires a deep appreciation of the branch of physics, we call quantum physics. However, as a first approximation, we can also interpret magnetism in its full bloom, with the help of classical physics.

Nature has given us materials with diverse magnetic properties. Ranging from the strongly magnetic iron and cobalt to the weakly magnetic (or in everyday language, the “non-magnetic”) rubber and water, there exists a remarkable variety of magnetic materials. The kings of these materials are the ferromagnets and ferrimagnets that are used in inductors, transformers, motors and generators, antennas, audio and video tapes, loudspeakers and microphones and the exotic gi-
ant magnetoresistance (GMR) devices. We can say that in electrical engineering, magnetic materials and devices [1] are as pervasive as oxygen! Without transformers with ferrite cores, for example, it would be almost impossible to have the vast electric grid and supply systems that have transformed the fate of the post-industrial man.

There are other weaker forms of magnetism as well, such as paramagnetism that pulls in the material towards a strong magnetic field. Paramagnetism plays a vital role in many important chemical processes such as catalysis. Furthermore, every material exhibits some form of diamagnetism, pushing the material away from the strong magnetic field. Superconductors are perfect diamagnets; they will repel strong permanent magnets to the extent that they will hover in mid-air when placed on top of a magnet. This principle of magnetic levitation is at the heart of the super-fast train, the MAGLEV.

The present experiment on “magnetic phase transitions” is quite different in character from our other experiments. The experiment is based on a reasoned and informed discussion about the apparatus and its various features. You will learn about new equipment, electric components and probe the safety features that have gone into the design of this setup.

You must follow all safety procedures and warnings. The experiment involves large electric currents and hot wires that can seriously risk your safety.

KEYWORDS

Ferromagnetism · Paramagnetism · Curie Point · Electrical Energy · Specific Heat Capacity · Stefan-Boltzmann Law · Variable Transformer · Digital Multimeter · Clamp meter · Electrical Safety

APPROXIMATE PERFORMANCE TIME 4 hours.

9.1 Conceptual Objectives

In this experiment, we will,

1. learn how to handle electricity, especially large currents, safely;
2. understand the role of thermal and electric insulators and conductors;
3. familiarize ourselves with common electrical test equipment such as the
voltmeter, ammeter, multimeter, clamp meter, tester and circuit components such as the circuit breaker and variable transformer (variac);

4. learn to interpret important thermal and electrical properties of materials;

5. appreciate the instantaneous nature of phase transitions; and finally,

6. appreciate the inter-relationship between electricity, magnetism and thermodynamics.

9.2 Experimental Objectives

The experiment determines the Curie point of a ferromagnetic material as its temperature is raised with the help of resistive heating. Besides the monitoring of the phase transition, we will also learn about the use of electrical measurement equipment and safety practices that must be observed when designing, operating or testing electric equipment.

9.3 Theoretical Introduction

Resistance and Ohm’s Law

Ohm’s law is an empirical observation deduced from experiments performed by Georg Simon Ohm in 1827. The modern form of Ohm’s law is,

\[ V = IR. \]  \hspace{1cm} (A-1)

where \( V \) is the applied potential in volts (V), \( I \) is the current in amperes (A) and \( R \) is the resistance in ohms (\( \Omega \)). Remember that Ohm’s law has limited validity and applies only to the so-called ohmic resistors.

Resistance is due to the interaction of moving charged carriers (electrons) and the fixed atoms or ions in a conductor. Due to the applied potential difference, moving charge carriers drift from higher to lower potential and in the process, collide with the atoms of the material. This results in increased vibrations of the atoms. The increased kinetic energy raises the temperature of the conductor because temperature is proportional to the kinetic energy. This is the origin of the heating effect of current, called Joule’s or resistive heating.
Chapter 9. Magnetic Phase Transitions of a Ferromagnetic Alloy

Power Dissipated

The electric power $P$ fed into any circuit component is given by,

$$P = VI.$$  \hfill (A-2)

If the component is a resistor, this power is dissipated as heat. Using Ohm’s expression (A-1), we obtain,

$$P = I^2 R.$$  \hfill (A-3)

We know that power is

$$P = \frac{\text{energy}(E)}{\text{time}(t)},$$ \hfill (A-4)

and substituting $P$ with (A-2) we obtain,

$$E = VIt.$$  \hfill (A-5)

Q 1. Suppose we have two heating elements with identical lengths, but one has a higher resistance than the other. Both are connected to identical voltage sources. Which of the elements will be heated more, the higher or the lower resistance element?

Q 2. Suppose we have two heating elements made from the same material, nichrome, commonly used in domestic heaters? One is twice the length of the other but the area is also doubled, so the two elements have the same resistance. They are connected to identical supplies for the same amount of time. Which wire would acquire the higher temperature?

Magnetism in Materials

To a very good first approximation, the origin of magnetism in materials lies in the motion of electrons. The magnetic material can be thought of as being composed of elementary magnets also called magnetic dipoles. These are similar to tiny magnets with a north and south pole. An atom contains electrons in motion. These electrons constitute a current and hence, produce a magnetic field. One such atom can be thought of as an elementary magnet. Now the material as a whole, will be made up of many elementary magnets. The arrangement and orientation of these elementary magnets determine the overall magnetic properties.
9.3. THEORETICAL INTRODUCTION

Paramagnetic materials

In paramagnetic materials, the elementary magnets are all randomly oriented. Suppose, we draw a tiny vector corresponding to the orientation of the dipole. Now take the vector sum of these dipoles. What do we get? The resultant is zero, showing that in the absence of an external field, the paramagnetic material is unmagnetized. However, this observation does not mean that there are no elementary magnets. The elementary dipoles still exist; it just happens that they completely cancel the effect of one another. This effect is shown in Figure 9.1(a).

Once we apply an external field with intensity $H$, the dipoles rotate and tend to orient in the direction of the field. This overall alignment results in a net magnetization $M$ of the sample. The alignment, however, is by no means perfect. At any temperature higher than absolute zero, the thermal agitation will kick the elementary magnets out of perfect alignment. The situation is clearly depicted in Figure 9.1. The dashed arrow shows the direction of the magnetization vector, that is defined as,

$$\vec{M} = \frac{\sum_k \vec{\mu}_k}{V},$$

(A-6)

where $\mu_k$ is the dipole moment of the elementary magnet and $V$ is the total volume of the sample [1, 2].

![Figure 9.1: The alignment of the elementary magnets in a paramagnetic sample. (a) Shows the situation when the applied field is zero. (b) As the applied field intensity $H$ is increased, the magnets preferentially tip in the direction of the applied field, resulting in a net magnetization of the sample.](image-url)
Ferromagnetic materials

On a microscopic level, approximately millionth of a meter, metals look like
drought struck soil of the summer sun. These ‘cracked segments’ are called
grains and the cracks are called grain boundaries (see page 54 of [2]). As the
name suggests, grain boundaries separate one grain from another.

Ferromagnetic materials are quite distinct in their character from paramagnetic
materials. Ferromagnets have regions called magnetic domains. Elementary mag-
nets within each domain are aligned with respect to one another, even though
the domains can be aligned in all possible directions.

Now, one grain can comprise more than one domains. Figure 9.2(a) is a simplified
representation of a polycrystalline material. The grain boundaries are shown as
dark lines whereas the domain walls inside the grains are drawn as thinner lines.
Within each domain, the net magnetization is represented by the dashed and dot-
dashed arrows. The domains are randomly oriented (Figure 9.2(c)). For the same
reason, even a strongly magnetic material such as iron can be unmagnetized in
the absence of a field.

As the applied field intensity $H$ is increased, the domains that are favourably
aligned, i.e., tilted towards the applied field, grow in size and the unfavourably
oriented domains shrink (Figure 9.2(d,e)). As the applied field is ramped up,
the growing domain engulfs the smaller domains with the result that there is
one domain per grain (Figure 9.2(f)). Finally, with a sufficiently strong field, the
magnetization of the grain (=magnetization of the domain) rotates so as to align
itself with the applied field.

Curie temperature

Ferromagnets have a much higher magnetization than paramagnets. In addition,
the phenomenon of ferromagnetism comes about due to a totally different mech-
anism. In ferromagnetic materials, the elementary magnets act in a cooperative
fashion, forcing neighbouring magnets to align within themselves. Soon all el-
ementary magnets within a domain are unitedly pointing in one direction. This
configuration lowers the energy, called the exchange energy.

The exchange energy, however, acts in conflict with the thermal energy that tends
to misalign the elementary magnets. As the temperature is increased, the thermal
energy starts dominating over the exchange energy and the magnetization drops.
Figure 9.2: The magnetic moments, domains and grains in a ferromagnetic material. (a) The grains and domains in a polycrystalline material. One grain comprises several domains and the magnetization within a domain is indicated by a dark arrow. (b) Domain microstructure of an amorphous ribbon (figure extracted from [3]). (c-g) Illustrations for a single grain. (c) The magnetization is zero in the absence of the applied field intensity $H$. (d-f) As the applied field increases, domains grow and shrink, to the extent that there is only one domain per grain, and (g) finally, the magnetization rotates in the direction of the applied field.

However, the material is still ferromagnetic as the domain structure is preserved. Above a critical temperature, the Curie temperature $T_c$, the ferromagnet suddenly turns into a paramagnet. The $T_c$'s of common ferromagnets are presented in Table 9.1.

<table>
<thead>
<tr>
<th>Element</th>
<th>$T_c$ (°C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fe</td>
<td>770</td>
</tr>
<tr>
<td>Ni</td>
<td>358</td>
</tr>
<tr>
<td>Co</td>
<td>1127</td>
</tr>
<tr>
<td>Gd</td>
<td>16</td>
</tr>
</tbody>
</table>

Table 9.1: Curie temperatures for common ferromagnetic elements [1].
9.4 Apparatus and Experimental Preparation

There are several examples of undergraduate experiments [4, 5, 6] used to determine the $T_c$ for various materials. The present experiment is an adaptation of the approach followed in [6].

A schematic sketch of the apparatus is shown in Figure 9.3 and photographs of some of the components are presented in Figure 9.4. Given below is a short description of the equipment used. You are required to note down answers to the following questions, before you start the experiment in the lab. Carefully observe the experimental setup and remember, "do not switch on the mains power supply".

Your experiment starts here!

![Diagram of experimental setup]

Figure 9.3: Schematic diagram of the experimental setup.

1. **Variac** The variac (*Electrodynamic Works, Karachi*) is a variable transformer. The ac mains supply from WAPDA (or the local generator) is connected across the primary coil and the variable output is taken from the sliding contact on the output side. The voltage is step down in the ratio of

$$\frac{V_{out}}{V_{in}} = \frac{N_{out}}{N_{in}} = \frac{I_{in}}{I_{out}}.$$  

(A-7)
Figure 9.4: Photographs of selected components: (a) variac, (b) digital multimeter, (c) clamp meter, (d) electric panel box (control box), (e) pole for ferromagnetic heating element.

where $N_{in}$ and $N_{out}$ are the total number of turns in the primary coil and the turns between the output tap and ground (Figure 9.5).

Figure 9.5: Simplified internal construction of a variac.
★ Q 3. The input of the variac is connected to the 220 V mains. The output is connected to a load resistor that draws a current of 5 A. If the ratio $N_{in}/N_{out} = 2/1$, what is (a) the output voltage and (b) the input current drawn from the mains?

★ Q 4. How is the variac different from an ordinary transformer?

2. **Digital multimeter** A digital multimeter (GW-Instek GDM-451) measures the output voltage from the variac.

3. **Clamp meter** Currents are measured with the help of a clamp meter (Kyoritsu). The jaws of the clamp meter surround the wire through which the current is to be determined.

★ Q 5. How does a clamp meter work? Will the clamp meter work for direct current (dc)?

★ Q 6. For alternating current (ac), the direction of the current is constantly changing, but the clamp meter shows one positive current reading. Resolve this apparent anomaly.

4. **Control box** The control box shown in Figure 9.6 has been designed and assembled in-house and serves as the main electric distribution box for the experiment. The panel is fitted with an analog voltmeter and ammeter that measure, respectively, the ac mains voltage and the current through the heating element. However, we will use the clamp meter for the most accurate current readings. The box is also fitted with a red emergency stop button. **Press this button in case of leakage of current or fear of electric shock.** The button can be reset by turning it clockwise and releasing.

The control box is also fitted with a circuit breaker (Terasaki) rated at 15 A. As soon as the current goes beyond the rated value, the circuit breaker trips and opens the circuit; the current drops to zero.

For electric protection of the circuit components, a magnetic contactor

![Figure 9.6: Internal wiring of the control box.](image)
(NHD Industrial Co., Taiwan, SC-16) has also been used. Ask your demonstrator if you want to know more about the working of the contactor.

The exposed metal parts of the apparatus, including the mounting screws of the control box, have all been earthed. This prevents electric shocks if by accident or damage, a live wire comes in contact with the metal body.

★ Q 7. What is the difference between a circuit breaker and a fuse?
★ Q 8. What is the function of the earth wire? Draw a simple diagram to describe your reasoning.
★ Q 9. What is the role of the magnetic contactor in the circuit?

5. **Ferromagnetic heating element** In our experiment, current passes through a ferromagnetic heating element. The element we have chosen is a commercially available material called Kanthal-D (Kanthal and Hyndman Industrial Products). We will use a heating element approximately 1.5 m in length and wound into a spiral shape. The spiral diameter is approximately 10 mm whereas the wire diameter is 0.6426 mm.

Some important properties of Kanthal-D relevant to this experiment are presented in Table 9.2.

<table>
<thead>
<tr>
<th>Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Composition</td>
<td>83.2%Fe 22%Cr 4.8%Al</td>
</tr>
<tr>
<td>Specific heat capacity c</td>
<td>460 J kg⁻¹ K⁻¹</td>
</tr>
<tr>
<td>Resistivity ρ at 20°C</td>
<td>1.39 Ωmm² m⁻¹</td>
</tr>
<tr>
<td>Emissivity ε</td>
<td>0.7</td>
</tr>
<tr>
<td>Density</td>
<td>7.25 g cm⁻³</td>
</tr>
<tr>
<td>Melting point</td>
<td>1500°C</td>
</tr>
</tbody>
</table>

Table 9.2: Important properties of Kanthal-D alloy [7].

★ Q 10. Define specific heat capacity and derive its units.
★ Q 11. Define resistivity and derive its units.
★ Q 12. Does the specific heat capacity change with temperature?

6. **Pole for Kanthal and magnet assembly**. The pole for the Kanthal and magnet assembly was fabricated locally (Noor Trading and Consultancy, Rawalpindi) and modified in-house. The Kanthal wire is hooked up between porcelain insulators fixed to the top and bottom arms (see Figure 9.4(e)). The middle arm has an array of ferrite disk magnets (Hall Road) epoxied
onto an alumina silicate base. The whole pole assembly is made of mild steel.

★ Q 14. Why have we used porcelain for mounting the heating element on the pole?

★ Q 15. Why have we used metal to construct this pole?

★ Q 16. Why cannot we attach the magnets directly to the metallic post? Why do we need to insert the alumina silicate ceramic in between?

★ Q 17. In the experiment we will use a tester to detect current leakage. The tester is made to touch a conductor. If the conductor is live, a small bulb inside the tester will glow. How does a tester work?

9.5 Experimental Method

Inspection

The mains supply is still switched off. Follow all rules and safety procedures. In this section, we will test the safety features of the circuit.

The electric current in this experiment can kill! Follow all rules.

★ Q 18. Visualize and sketch a possible circuit diagram for the experiment?

Q 19. Check the zero error for the analog voltmeter and ammeter on the control box. Use the adjusting screw at the base of the pointer to correct.

★ Q 20. Attach a three-pin shoe to the control box.

★ Q 21. What are the different colour codes for wires in live, neutral, earth? Suggest why do we have two additional colours.

Ask the demonstrator to check the electrical connections. Do not attempt to switch on the mains supply in the absence of the demonstrator.
9.5. **EXPERIMENTAL METHOD**

**Q 22.** Set the regulator on the variac to its minimum output voltage, zero. The demonstrator will switch on the mains supply. Press the green START button on the control box.

**Q 23.** Check that there is no current leakage using a tester in all three components (variac, control box and pole). Put the tester on bare metal surface to check for leakage.

**Q 24.** Slowly increase the output voltage (voltage from the output of the variac) to 15 V (measured on the digital multimeter). Check for current leakage again.

★ **Q 25.** What is the reading on the clamp meter?

★ **Q 26.** Test the emergency stop button. Does the clamp meter reading go to zero? If it doesn’t, immediately inform the demonstrator.

★ **Q 27.** Set the regulator on the variac to its minimum output voltage, zero, again and press the green START button.

★ **Q 28.** Test the circuit breaker, using the test button.

★ **Q 29.** Hook the clamp meter to the WAPDA mains and measure the current.

**Measurement of Curie temperature**

★ **Q 30.** Set the output voltage regulator on the variac to 24 V and press the green START button.

★ **Q 31.** Measure the current (using the clamp meter) and the time it takes the wire to reach the Curie temperature. You will be provided with a stopwatch. When the heating element snaps away from the magnet, immediately press the red STOP button to switch off the circuit.

★ **Q 32.** Repeat the experiment twice or thrice at one voltage setting. Allow the heating element to sufficiently cool between two successive measurements.

★ **Q 33.** Repeat the experiment nearly five times, always keeping the voltage below 35 V.

★ **Q 34.** Switch off the mains supply.
Calculations

The electrical energy supplied in a certain interval of time is defined in Equation (A-5). In the present experiment, this energy is used up in two processes:

1. absorbed by the heating element, raising its temperature from the ambient room temperature $T_0$ to the Curie temperature $T_c$; and

2. radiated away by the heating element.

The energy absorbed $E_a$ may be expressed as

$$E_a = mc(T_c - T_0) \quad (A-8)$$

where $m$ is the mass of the wire and $c$ the specific heat capacity. The mass can be measured using the provided weigh balance.

The energy radiated ($E_r$) form the wire is,

$$E_r = \varepsilon \sigma S(T_c^4 - T_0^4) t \quad (A-9)$$

where $T_c$ is the Curie temperature, $\varepsilon$ is the emissivity, $\sigma$ is the Stefan-Boltzman constant ($\sigma = 5.675 \times 10^{-8}W/m^2K^4$) and $S$ is the surface area of the heating element.

**Q 35.** Using Equations (A-5),(A-8) and (A-9) and the principle of energy conservation, write down the energy balance equation.

**Q 36.** Express your final equation in terms of the data obtained in the previous section. This step requires some careful thinking. You will obtain an equation with the unknown variable $T_c$.

**Q 37.** Run Matlab on the PC and solve the energy balance equation using the command,

```matlab
>> solve('equation')
```

where `equation` is inserted within single quotes and represents the Matlab format for the energy balance equation.

**Q 38.** What are the four different numbers that you see? Which one will you choose?

**Q 39.** Convert your answer to degrees Celsius. What is the $T_c$ for the Kanthal-D alloy?
9.6 Experience Questions

1. While using a tester, does the current really pass through us?

2. Why do our muscles jerk when we get an electric shock?

3. Can high magnetic fields affect our nervous system?

9.7 Idea Experiments

1. What is the Hall effect? Use a Hall effect sensor to measure the magnetic field due to permanent magnets of various shapes and strengths [4].

2. Ferromagnetic materials display the phenomenon of hysteresis. Build a teaching apparatus that demonstrates hysteresis [9].

3. Drive an electric bulb at various frequencies. Plot the voltage-current relationship and see whether the resistive element in the bulb satisfies Ohm’s law. Furthermore, demonstrate hysteresis in the behaviour of the resistive element [10].
Bibliography


Chapter 10

Optical Activity of the Chiral Solutions

Sabeeh Anwar and Wasif Zia

Visible light is only a small fraction of the electromagnetic spectrum. In fact, light is composed of oscillating electric (E) and magnetic field (B) vectors. These vectors are mutually perpendicular as well as perpendicular to the direction of the propagation.

Light passing through vacuum is the simpler to understand, moving unperturbed with the speed $c \approx 3 \times 10^8$ m/s but in the present experiment we would like to understand the more complicated (and aesthetically assuring!) problem of how radiation interacts with matter. The simple experimental setup will help us appreciate one facet of light’s wave nature itself, its polarization and how this polarization can rotate as it traverses an optically active liquid.

KEYWORDS

Polarization • Optical Activity • Chirality • Stereochemistry • Enantiomers • Laser • Photodetector • Cathode Ray Oscilloscope

APPROXIMATE PERFORMANCE TIME 4 hours.
10.1 Conceptual Objectives

In this experiment, we will,

1. learn how to use lasers safely;

2. practice aligning of optical setups;

3. investigate the concept of standard errors in simple measurements;

4. learn the use of cathode ray oscilloscopes (the work-horse of most science laboratories);

5. identify uncertainties in measurement of angular and linear scales;

6. correlate experimental data with mathematical expressions;

7. observe the difference between accurate and precise results;

8. learn that for wave motion, intensities are squares of amplitudes; and

9. reveal the inter-disciplinary character of experimental science, revisiting an experiment that is routinely performed in the chemistry laboratory.

10.2 Experimental Objectives

The experiment manifests the wave nature of light through the concept of polarization. We will also find the optical activity of a commonly available compound showing how the plane of polarization can be rotated by certain molecules. Finally, we will demonstrate the concept of “optimal sampling”.

10.3 Theoretical Introduction

Polarization

Light is emitted by tiny atomic or molecular emitters. The electric field vector from a single emitter oscillates in a plane perpendicular to the direction of propagation. The light is said to be linearly polarized. We may have horizontally (H) or vertically (V) polarized light as shown in Figure 10.1, where the electric field vector vibrates in the horizontal or vertical planes, respectively. We may also have polarizations that are titled with respect to these axes, 40°, 60°, 230° and so on.
Most objects we are familiar with, however, emit unpolarized light. The planes of polarization are oriented randomly in all possible directions. A polarizer picks up only plane of vibration from all these random orientations and produces plane polarized light. Materials that have this property are generally used in making polarized sunglasses.

The direction of the plane of polarization emerging from the polarizer depends on its optical axis. We use a polarizer in which the direction of polarization is marked by two diametrically opposed green spots. (However, you cannot see them because the lab demonstrators have already fitted the polarizers into their mounts.) Suppose we orient the optical axis so that we obtain vertically polarized light. The electric field vector is always pointing along the vertical (±\(\hat{z}\)) direction while the light moves forward in the horizontal \(\hat{y}\) direction. For such an arrangement, the electric field is,

\[
E_0 \cos \left( k y - \omega t + \phi \right) \hat{z},
\]

where \(E_0\) is the amplitude, \(\omega = 2\pi f\) is the frequency, \(k = 2\pi / \lambda\) is the wave number and \(\phi\) is the phase of the wave. Make sure you understand these terms [1].

On the other hand, if the optical axis were horizontal (along the \(\hat{x}\) axis), we would obtain horizontally polarized light,

\[
E_0 \cos \left( k y - \omega t + \phi \right) \hat{x}.
\]

The direction in which light propagates is still along the \(\hat{y}\) axis. Note that the unit vector written in bold denotes the direction of the plane of polarization, whereas the variable \(y\) next to \(k\) denotes the direction of propagation.

We can also get circularly polarized light. You will learn more about these concepts in physics lectures. In this case, the field vector rotates along a helix just...
like a screw. When viewed from the end, the vector is in fact moving in a circle. This happens when you add or subtract two components that are 90° out of phase, a sine and a cosine. This circularity as shown in Figure 10.2 can be right or left depending on how you add or subtract the components.

\[ E_0 [\cos (ky - \omega t + \phi_0) \hat{z} - \sin (ky - \omega t + \phi_0) \hat{x}], \]  

(A-3)

\[ E_0 [\cos (ky - \omega t + \phi_0) \hat{z} + \sin (ky - \omega t + \phi_0) \hat{x}], \]  

(A-4)

Figure 10.2: Right and left circularly polarized light.

The tip of the electric field vector rotates clockwise or anticlockwise for circularly polarized light, but the important thing to remember is that the phase (\(\phi_0\)) is the same for both the components. The accompanying Figures may help you perceive this concept better.

**Q 1.** Write down the mathematical expression for the electric field for light that is linearly polarized with a plane of polarization making an angle of 45° with the \(\hat{x}\) and \(\hat{z}\) axes? In which direction is the light propagating?

**Malus's Principle**

In our experiment randomly polarized light from a laser (of wavelength 633 nm) passes through two polarizers labelled A and B as shown in Figure 10.4. The field emerging from A is given by the expression (A-1). The field emerging from B is determined by the relative orientations of the optical axes of A and B. The polarizer B used in this way is called an analyzer.

If the relative orientation between A and B is \(\theta\) as shown in the Figure 10.3, then the emergent electric field after passing through B becomes,

\[ E_0 \cos \theta \cos (ky - \omega t + \phi) \hat{z}, \]  

(A-5)
10.3. THEORETICAL INTRODUCTION

Figure 10.3: Orientation of B.

This represents a decrease in the amplitude by a factor of \( \cos \theta \). But experimentally, we measure the intensity, not the electric field. As the intensity is proportional to the amplitude squared, the reduction factor is \( \cos^2 \theta \). This is a statement of Malus's principle, mathematically written as

\[
I_B = I_A \cos^2 \theta.
\]

We will measure intensities using a silicon photodetector. Remember that \( I_A \) and \( I_B \) are the intensities of the radiation emerging from the polarizers A and B.

![Schematic sketch of the experimental setup.](image)

Figure 10.4: Schematic sketch of the experimental setup.

In this sense, the polarizers act as “projectors”—they project a vector onto a certain axis. It is also possible to do something more complicated, i.e., to physically rotate the plane of polarization. This is achieved through optical components called wave plates or retarders. In our experiment we will achieve the same effect using certain chemical substances that are called optically active substances. What are the basic properties of optically active substances and why are they important? This question requires a brief digression into the area of stereochemistry.

**Chirality and Enantiomers**

Try to recall what you saw when you were combing your hair in the morning or brushing your teeth for that matter. What did you see? “Your image!” That is such a trivial question. But what if you are asked to place your image on top of
your original self. Will they coincide or not? The answer is 'No! They will not'. This 'placing on top' is called superposing. If you have parted your hair on the right, your image has parted them on the left. Try shaking hands with your mirror image. You will be confounded. Similarly there is no painless way of wearing your left shoe on your right foot.

With this background, consider a molecule which is a network of atoms arranged in three dimensional space. Two molecules may have an identical composition of atoms and the same bonding network, however they may still differ in their detailed three dimensional arrangement. Surprisingly, these variants of molecules can have totally different physical and even chemical properties.

For example, consider bromochlorofluoromethane. Its structure is shown in Figure 10.5. Now place a mirror next to the molecule and observe the image. The mirror image has a distinct configurational arrangement and cannot be superposed onto the original molecule (without of course, reflection through the mirror plane). The molecule and its non-superposable mirror image are enantiomers of each other and the corresponding property is called chirality. Only chiral molecules are optically active.

![Bromochlorofluoromethane](image)

Figure 10.5: (a) Structure of bromochlorofluoromethane and (b) its ball-and-stick model shown alongside the mirror image. The wedge shaped arrows in (a) represent chemical bonds pointing into or out of the plane of the paper.

**Q 2.** Identify the chiral molecules: (a) 3-methylhexane (b) 3-methylpentane (c) the amino acid glycine and (d) dibromochlorofluoromethane. The structures are shown in Figure 10.6.
Chirality is seen throughout the biological world. With the exception of inorganic salts and a few low molecular weight organic substances, molecules in living systems, both plant and animal, are generally chiral. For example, only one of the stereoisomers called (S)-alanine occurs naturally. Enzymes that catalyze biochemical reactions are also highly stereoselective, i.e., they will speed up reactions only with one enantiomer of the chiral pair. Chymotrypsin, a chiral intestinal enzyme will break down only the corresponding peptide enantiomer during digestion. This 'chiral favouritism' in nature is one of the open questions in the life sciences. Researchers have even linked this with the similar question in cosmology of why the universe is made up of matter and not of antimatter!

In the chemical industry too, there is a drive towards synthesizing chiral catalysts for developing stereoselective reactions. Chirality is also an important factor in drug efficacy and design. Drugs that are packaged as racemic mixtures, comprise equal amounts of the two enantiomers. However in most cases, only one of these molecules is biologically active. For example in ibuprofen, sold as an analgesic in Pakistan in the racemic form, only the (S)-enantiomer is active [6]. Chiral drugs have now become a focus of most pharmaceutical companies. For example Naproxen available in this country is a chiral molecule and is sold in the enantiomerically pure form. Statistics show that about 56 percent of the drugs in present use are chiral molecules [7].
Light as a Chiral Probe

The human foot is a chiral entity. That is why a right shoe cannot fit on the left foot and a left shoe cannot fit on the right foot. However, the traditional footwear, the khussa can—as it cannot distinguish between “leftness” and “rightness”. The shoe is a chiral probe whereas the khussa is an achiral entity.

Analogously, it is impossible to physically distinguish between two enantiomers using unpolarized light. One needs plane polarized light to test chirality.

The experiment that performs this probing is in fact, very simple. Shine polarized light onto an optically active substance. The plane of polarization rotates in one direction or the other. (We cannot tell before hand, the direction of physical rotation. This has to be determined from experiment.) For example, the chiral molecule d-glucose bends light to the right (when viewed along the direction of propagation) and this is experimentally determined. The prefixes d and l signify ‘dextro’ (right) and ‘levo’ (left) physical rotations.

Q 3. Why is plane polarized light chiral? Why is randomly polarized light achiral? HINT: Draw a one-sided arrow pointing upwards representing plane polarized light and reflect it across a plane perpendicular to the arrow.

Q 4. Polarized light is shone through a racemic mixture of glucose. In which direction will the plane of polarization rotate?

Q 5. A mixture of l-2-butanol and d-2-butanol rotates the polarization plane in the left direction through 10°. If pure l-2-butanol has a rotation of 13.5° in the same direction, determine the composition of the mixture.

Q 6. A liquid is made up of molecules randomly jostling in all directions. Investigate why this randomly oriented jumble-up can, in fact, rotate light so coherently in one direction. Note: This is a tough question!

In a chiral medium, each molecule contributes to the optical rotation. More molecules would imply stronger rotation. So a longer path length and a more concentrated solution would result in greater rotation angles. For comparison, we often normalize with respect to the length of the sample and concentration, resulting in the specific optical activity,

$$\theta = \frac{\theta}{c}$$

(A-7)

where c is the concentration and l is the path length. It is also important to mention the temperature and wavelength of the light used.
Figure 10.7: Photograph of the experimental setup. Refer to the Section on Apparatus for a description of the components.

Q 7. A sample of an optically pure enantiomer yields an optical rotation of about 180°. How will you determine whether it is dextrorotatory or levorotatory?

Q 8. What are the units of [θ]?

10.4 Apparatus

The schematic of the experimental setup is shown in Figure 12.7 and a digital photograph in Figure 10.7.

1. Laser The source of light in the experiment is a continuous-wave He-Ne laser (Melles-Griot 25-LHR-073 or Thorlabs HRR020) with a wavelength of 633 nm (red) and output power of 2 mW. The output from the laser is randomly polarized. The laser is mounted on a V-shaped housing (Thorlabs) attached to a mounting post (Thorlabs).

Lasers can be very dangerous if mishandled or if the relevant safety procedures are not followed. Always contact the lab staff if you have any doubt. In all cases, it is incumbent that you always abide by these safety precautions for our Class-IIIa laser.

- Never look directly into the laser beam or direct the beam to anyone else or to an area where people are present. This may result in serious eye damage.
- Wear the properly rated safety glasses (Thorlabs) while performing
the experiment.

- Do not scoop down to the level of the table or bring the laser to the level of the eye.

- Do not disconnect or connect the laser head from its high voltage power supply. The demonstrators have already completed this step for you.

- Do not remove the laser head from its housing.

2. Polarizers and rotation mounts The experiment uses two polarizers (Thorlabs LPVIS050) at the positions A and B in the experiment. These polarizers are mounted and retained in rotation mounts (Thorlabs RSP05/M). Be careful not to touch the surface of the polarizer. This will scratch the surface resulting in permanent damage. The rotation mount can be locked and unlocked with a key that is provided with the setup. The rotation mount is provided with an angular scale, that keeps track of the polarizer orientations.

3. Optical activity cell This is a cylindrical glass cell that has been designed in-house and manufactured by the glass-blowers (Marghob Scientific Store, Lahore) in the market. The cell is supported by two crescent-shaped holders and posts manufactured locally (Crown Engineering Works, Lahore). The cell is fitted with inlet and outlet ports. When you start the experiment, the demonstrator will have already filled up the cell for you.

4. Silicon photodetector The photodetector (Newport 818-SL) converts incident light intensity into current. Ensure that the light falls in the central region of the photodetector. This is where the sensitivity (output current divided by the incident intensity) is flat and maximum. The output current is fed into an oscilloscope where it is converted into voltage.

5. Cathode ray oscilloscope The cathode ray oscilloscope (GW-Instek GOS-635G) is a ubiquitous tool used in most science laboratories, especially in physics and electrical engineering. A familiarity with this instrument is crucial, as it is likely to remain attached to your academic life at the SSE. In the present experiment, we will use the oscilloscope to measure voltages.
10.5 Experimental Method

Preparation

The laser has already been turned on by the demonstrators. Note that each time the laser is switched on, it takes about 15 minutes to warm up and reach a stable intensity.

★ Q 9. The demonstrator has already closed the shutter of the laser. Now connect the detector to the oscilloscope. Turn the input mode to GND. You should see a straight line. Turn the vertical positioning knob of the channel to place the line on the middle of the screen. Change the input mode to DC. Does the vertical position of the line shift? If yes, how much? This is background reading from ambient light? Turn the vertical positioning knob so that the background level is at the datum. All intensity readings will now be referenced to the background.

★ Q 10. Ask the demonstrator to open the laserhead shutter. Align the detector so that the laser spot falls on the approximate middle of the detector. What is the reading on the oscilloscope? Call it $I_0$. What is the voltage sensitivity setting on the scope? What are the units of $I_0$?

★ Q 11. Place the polarizer A near the output of the laser. Adjust its height and orientation and align the polarizer in the path of the optical beam. Call the intensity measurement with A as $I_A$? Why is $I_A < I_0$?

★ Q 12. Unscrew the lock on the top of A using the hex key provided. Now rotate the optical axis of the polarizer through large angles on both sides. Does the reading $I_A$ vary? Describe your observation.

★ Q 13. Now place the second polarizer B in front of the detector. The intensity recorded on the scope will change to $I_B$? How does $I_B$ compare with $I_A$? Unscrew the lock on top of B. What happens to $I_B$ as you rotate B through large angles?

Malus's principle

★ Q 14. Note the reading on the rotation mount A. Call it $\alpha$. What is the uncertainty in $\alpha$?
**Q 15.** Slowly rotate B so that the intensity is maximized? Note down the value of the angular position of B, calling it $\beta$. Note down the corresponding intensity $I_B$.

**Q 16.** Rotate $\beta$ in steps of 20°, keeping all rotations clockwise or anticlockwise. Take approximately 20 readings. At each step, record $I_B$. Keep $\alpha$ fixed throughout.

**Q 17.** Plot the values of $I_B$ versus $\beta$.

**Q 18.** Fit your data to a suitable function. What is the fitting function and the goodness of the fit? Describe your observations in light of Malus's principle.

**Determination of the optical activity**

**Q 19.** Note down the value of $\alpha$, the reading on the polarizer A.

**Q 20.** Adjust $\beta$ on the polarizer B such that the intensity is maximized. Call this intensity $I_{\text{max}}$. Note down the value $I_{\text{max}}$ as well as the value of the corresponding $\beta$, calling it $\beta_{\text{max}}$.

**Q 21.** Now rotate $\beta$ such that the intensity is minimized. The minimum intensity is called $I_{\text{min}}$ and the corresponding angle is $\beta_{\text{min}}$. What is the angular difference between $\beta_{\text{max}}$ and $\beta_{\text{min}}$? Note down the values of $I_{\text{min}}$ and $\beta_{\text{min}}$.

**Q 22.** Calculate the intensity half way between $I_{\text{max}}$ and $I_{\text{min}}$.

$$I_{1/2} = I_{\text{min}} + \frac{I_{\text{max}} - I_{\text{min}}}{2} = \frac{I_{\text{max}} + I_{\text{min}}}{2}. \quad \text{(A-8)}$$

Adjust $\beta$ to locate the point where the intensity equals $I_{1/2}$. Call this angle $\beta_{1/2}$. Note down the values of $I_{1/2}$ and $\beta_{1/2}$.

**Q 23.** Now vary the angle $\alpha$ by some fixed amount, say 20° and repeat the procedure of finding the maximum, minimum and average intensities, noting down the intensities and corresponding angles. Take approximately ten readings, tabulating your results in the suggested format (Table 1).

<table>
<thead>
<tr>
<th>Sr.No.</th>
<th>$\alpha$ (deg.)</th>
<th>$\beta_{\text{max}}$ (deg.)</th>
<th>$I_{\text{max}}$ (V)</th>
<th>$\beta_{\text{min}}$ (deg.)</th>
<th>$I_{\text{min}}$ (V)</th>
<th>$\beta_{1/2}$ (deg.)</th>
<th>$I_{1/2}$ (V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>0</td>
<td>76</td>
<td>0.22</td>
<td>166</td>
<td>0</td>
<td>122</td>
<td>0.11</td>
</tr>
<tr>
<td>2.</td>
<td>20</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 10.1: Suggested format for tabulating the experimental results.
10.5. EXPERIMENTAL METHOD

★ Q 24. The demonstrators have already prepared a sucrose solution of a known concentration and filled it into a glass cell. Perch this cell on the posts and place it in between the polarizers A and B. The laser light must enter and emerge from approximately the middle of the side ends of the cell.

★ Q 25. Construct another table similar to the one above. This time find the maximum, minimum and average intensities and the corresponding angles with the optical cell in place.

★ Q 26. Once you have finished taking all the necessary readings, the demonstrator will switch off the laser power supply.

Data analysis for determination of the optical activity

All the remaining steps can be performed on paper or on the PC. From the available data, you are required to determine the optical activity of the solution.

★ Q 27. What is the optical activity based on the maxima of the intensity? What is the standard error in the measurement?

★ Q 28. What is the optical activity based on the minima of the intensity? What is the standard error in the measurement?

★ Q 29. Are the two results statistically different or the same? Justify your answer.

Figure 10.8: Maximum and minimum.

★ Q 30. Suppose you are given data that resembles the square of a cosine curve, as in (A-1). You measure the data along the curve. In which of the regions would one expect to get more precise results? Considering the accompanying Figure, would it be near the peak (a), the trough (b) or half way in between (c)? HINT: Think in terms of the slope of the cosine squared curve.

★ Q 31. Finally, determine the optical activity based on the intensities that are the average of the minima and maxima? What is the standard error in the
measurement? Is this a more accurate result? Is this a more precise result?

★ Q 32. What are the main results of this experiment.

10.6 Experience Questions

1. Does the sun emit polarized light?

2. Can a star emit polarized light? If yes / no, what consequences could be seen in nature?

3. Is the light from the sky polarized?

4. A small radio receiver is placed near a radio transmitter. What is the optimum orientation between the transmitter and the receiver that maximizes the reception?

5. At the time of dusk, the atmosphere appears somber blue but looking directly at the sun, it has an orange or pinkish tinge. Reason why.

10.7 Idea Experiments

1. Find the optical activity of kerosene or octane.

2. Does the intensity of light change the measurement of optical activity?

3. Measure the birefringence of liquid crystals [9].
Bibliography


Chapter 11

Data Acquisition and Filter Design

Umer Hassan and Sabieh Anwar

The experiment is an introduction to electronics. Our main objective in this experiment is to filter out noise in electric circuits. This experiment is divided into sections, such that each section introduces one of the key concepts, and finally this culminates to our final objective.

KEYWORDS

Breadboard · Data Acquisition · Sampling · Signal reconstruction · Nyquist Theorem · Logic Gates · Filters · Noise · Frequency Spectrum · Composite Signal

APPROXIMATE PERFORMANCE TIME 4 hours

11.1 Conceptual Objectives

In this experiment, we will,

1. learn how to implement circuits,

2. practice data acquisition,

3. understand the inter-relationship between mathematical expressions and graphs,
4. learn the use of electric test and measurement equipment.

5. appreciate the concept of frequencies.

6. observe the conversion between analog and digital signals, and

7. practice how to extract useful information from graphs.

11.2 Experimental Objectives

The experimental objectives are presented at the head of each Section. There are nine sections comprising this manual.

1. Introduction to the history of electronics

2. Breadboard layout and its internal connections

3. Data acquisition system

4. Understanding the frequency concept

5. Verifying the Nyquist theorem

6. Understanding logic gates exemplified by an XOR gate

7. Filter design

8. Learning frequency composition of a signal

9. Understanding the composite signal

10. Filtering a composite signal

11.3 Introduction to the History of Electronics

The history of modern electronics can be traced back to 1883, when Edison discovered that electrons flow from one metal conductor to another through vacuum. This is known as thermionic emission.

In 1987, J.J. Thomson developed a vacuum tube to carefully investigate the nature of cathode rays. He showed that the cathode rays were made up of particles, which he named "corpuscles". This marked the discovery of the electron. Thomson received the Nobel Prize in Physics in 1906.
CHAPTER 11. DATA ACQUISITION AND FILTER DESIGN

In 1904, John Fleming applied Edison's thermionic emission to invent a two-element electron tube called the diode. This was followed by Lee De Forest's discovery, in 1906, of the three-element tube, the triode. These vacuum tubes made possible the amplification and transmission of electrical signals.

In 1947, John Bardeen and Walter Brattain, working at Bell Telephone Laboratories, were trying to understand the nature of the electrons at the interface between a metal and a semiconductor. They realized that by making two point contacts very close to one another, they could make a three terminal device called the transistor, the semiconducting analog of the triode.

![Figure 11.1: John Bardeen, William Shockley, and Walter Brattain.](image)

The invention of the transistor, initiated the electronics revolution of the twentieth century. The drive was to build more transistors on a single chip. In 1965, Gordon Moore, co-founder of Intel, observed that the number of transistors per square inch on integrated circuits had doubled every year since the invention of the integrated circuit. Moore predicted that this trend would continue for the foreseeable future. In subsequent years, the pace slowed down a bit, but data density has still doubled approximately every 18 months.

Today, as the trend and need towards miniaturization is gaining momentum, there is also a growing realization that the physical limits of the transistor fabrication have been achieved. So, new fields have now emerged, such as quantum computing, spintronics, nanoelectronics, and so on.
11.4 Breadboard Layout and its Internal Connections

Objective

The objective of this section is to familiarize you with the internal connections of the breadboard.

Breadboard

A breadboard is used to make up temporary circuits for testing or to trying out an idea. No soldering is required, so it is easy to change connections and replace electronic components.

![Breadboard Image]

Figure 11.2: Breadboard.

Internal connections

Figure 11.3 shows the layout and internal connections of the breadboard. The holes in black are used for inserting the electronic components. The line joining holes shows their serial connection.

IC placement on the breadboard

Figure 11.4 shows how to place an integrated circuit (IC) chip on a breadboard.
11.5 Data Acquisition System

Objective

This section describes the experimental setup, particularly the data acquisition system, which is being used in different activities of this experiment.

Experimental layout

The basic layout of the Data Acquisition System for our experiment is shown in Figure 11.5.

1. Signal source

The signal is provided by the signal generator (GW-INSTEK). Sine wave,
square wave and triangular shaped signals can be generated using the signal generator.

2. **Signal routing**

   The signal routing and conditioning module (*National Instruments SCC-68*) is used for signal routing. The signal is taken from the breadboard using the provided hook-up wires. The SCC-68 module is shown in Figure 11.6.

   ![NI SCC-68 Module](image)

3. **Cables and accessories**

   The SCC-68 module is provided with the cable. It consists of a male connector which is further attached to the DAQ card.

4. **DAQ card**

   The DAQ card (*NI PCCl 6221*), shown in Figure 11.7 acquires the signal. If the incoming signal is analog, then DAQ performs *analog to digital conversion*. This is done using a chip called the *analog to digital converter* (ADC). The ADC is located on the DAQ card.

   For analog to digital conversion we have to *sample* the incoming signal. The *sampling frequency* must be greater than twice the fundamental frequency of the incoming signal. This criterion is stated in the **Nyquist Theorem**. Sampling and the Nyquist theorem will be discussed further in Section 7.
5. **LabVIEW**

The DAQ system is controlled through LabVIEW software. We are using LabVIEW version 8.5.1 on our computers.

### 11.6 Understanding the Frequency Concept

**Objective**

This section gives insight into the relationship between frequency and time period.

**Definitions**

1. **Frequency**

   It is a measure of the number of occurrences of a repeating event per unit time. It is usually denoted by $f$ and its units are Hertz (Hz) or cycles per second.

2. **Time period**

   It is defined as the time required for a single cycle in a repeating event. It is denoted by $T$ and its SI units are seconds. So we notice that frequency is the reciprocal of the time period. Mathematically, this is expressed as,

   $$ f = \frac{1}{T}. $$

   (A-1)
11.7. VERIFYING THE NYQUIST THEOREM

Example

Figure 11.8 shows the two sine waves. Figure 8(a) shows a sine wave with time period of 6 ms, whereas Figure 8(b) shows a sine wave whose time period is 3 ms. Thus the second sine wave has half the time period but double the frequency.

![Figure 11.8: Sine Waves.](image)

11.7 Verifying the Nyquist Theorem

Objective

This section describes some of the key concepts in data acquisition and signal processing—sampling and analog-to-digital conversion.

Sampling

Sampling a signal means to take the value of signal at discrete intervals of time, thus converting a continuous signal into a discrete signal.

Sampling rate is defined as the number of samples taken per second.
Example

Figure 11.9(a) shows a simple continuous sine wave with a time period of 6 ms. Figure 11.9(b) shows the digitized, sampled version of this signal, with a sampling rate of 5 KHz or a sampling interval of 200 μs.

Figure 11.9: (a) Continuous time analog sine wave and (b) its sampled version.

An analog input signal is fed into the DAQ card. The ADC on the DAQ card samples the analog voltage and stores the sampled points in the memory. The sampled points will have an amplitude distribution resembling the one in Figure 11.9(b).

Block diagram of an ADC

The functional block diagram of an ADC is shown in Figure 11.10. It takes the analog signal as an input and returns the sampled signal.

Curve fitting

After sampling, it might be required to fit the samples to a mathematical function. This could be done by using the curve fitting techniques, which we have learnt in the Matlab tutorials. Figure 11.11 shows how the acquired samples are fitted
using the Matlab command \texttt{lsqcurvefit} that employs the least square curve fitting algorithm.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure11.11.png}
\caption{Least Square curve fitted waveform.}
\end{figure}

\textbf{Signal reconstruction}

The result of data acquisition is an array of sampled points. The analog, continuous nature of the signal has been lost. After the process of sampling, the resulting sample values can be used for re-constructing the original signal. But, we have to determine the appropriate sampling rate that allows for reconstruction. If the sampling rate is not high enough, it is impossible to get the original signal back. This is the essence of the \textit{Nyquist Theorem}.

\textbf{Nyquist sampling theorem}

The theorem states that for the perfect reconstruction of a signal the sampling frequency must be greater than twice the maximum frequency of the signal being
sample.

**Q 1.** Can you guess what is the minimum number of samples per second required to reconstruct the 1 Hz sine wave?

**Experiment**

We now use our DAQ system to generate and acquire a signal and sample it at different rates. We observe the appearance of the corresponding reconstructed signal and determine the frequency of the reconstructed signal. We will comment on the accuracy of the reconstruction process. Thus, we will experimentally verify the Nyquist Theorem, and of course, in the process we also learn about sampling.

**Procedure**

**Q 2.** Carry out the following procedure.

1. The DAQ System has already been setup for you.

2. Familiarize yourself with the apparatus.

3. Generate a sine wave as an input signal using the signal generator.

4. Set the output frequency on the signal generator to 10 Hz.

5. Run the *nyquist.vi* file.

6. In the *Block Diagram* window, enter the *rate*. It is the sampling rate, the rate at which ADC samples the input signal.

7. In the *front panel* window click the *Run* button (shown by the arrow key).

8. Data acquisition starts, the output starts appearing on the waveform graph.

9. Samples are stored in *nyq.km* file.

10. Now start *Matlab*

11. In the *Matlab* window, type.

        >> nyquist;

12. This command asks you to input the sampling rate, which then returns the reconstructed waveform.
13. Now observe the outputs and complete the Table 11.1 on your note book, with sampling rates of 3, 4, 8, 15, 20, 30, 50, 70, 100, 500, 1000 Hz.

<table>
<thead>
<tr>
<th>Sampling Rate (Samples per second or Hz)</th>
<th>Sketch the waveform</th>
<th>Frequency of the reconstructed waveform (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 11.1: Illustrating Nyquist Theorem.

★ Q 3. Write your observations and inferences on your note books.

11.8 Logic Gates Exemplified by the XOR Gate

Objective

This section introduces you to digital logic, integrated circuit chips and truth tables. It also offers more practice in using a DAQ system.

Logic levels

In the world of digital logic, we always come across the terms logic level high and logic level low. In binary digital logic, we have two voltage levels only, high and low. Logic level high is considered to be approximately +5 V whereas logic level low is equal to 0 V approximately. The logic levels for an arbitrary binary digital waveform (TTL) are shown in Figure 11.12.

Figure 11.12: Logic level representation.
Logic gates

The components that inter-convert signals between logic levels are called logic gates. The basic logic gates include AND, OR, NOT, NAND, XOR, and NOR gates. These gates are used for implementing a variety of logic circuitry in digital systems. Logic gates lie at the heart of all computers. In this section we are going to study an XOR gate.

PIN Configuration

Figure 11.13 shows the PIN Configuration of the IC 7486, which is a quad XOR gate IC. Quad means that the chip contains four XOR gates.

![PIN Configuration of 7486](image)

Logic Diagram

Figure 11.14 shows the logic diagram of the 7486 IC. This illustrates the four XOR gates, that are fabricated in a single chip. The Figure also shows the pin number associated to each XOR gate.

![Logic Diagram of IC 7486](image)
Function Table

As shown in Table 11.2, each of the XOR gates has two inputs and one output. For example, the topmost gate has inputs D0a and D0b and output Q0. Table 11.2 shows the function table, also called the truth table of an XOR gate. The table specifies the output of the gate for the specified inputs.

<table>
<thead>
<tr>
<th>Input (Dna)</th>
<th>Input (Db)</th>
<th>Output (Qn)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
<td>L</td>
<td>L</td>
</tr>
<tr>
<td>L</td>
<td>H</td>
<td>H</td>
</tr>
<tr>
<td>H</td>
<td>L</td>
<td>H</td>
</tr>
<tr>
<td>H</td>
<td>H</td>
<td>L</td>
</tr>
</tbody>
</table>

Table 11.2: Function Table of an XOR gate.

Data Sheet

Every manufacturer of an IC sends the documentation for the IC, which is called the Data Sheet. It contains all necessary information about the IC. It also includes a function table, pin configuration and logic diagram of the IC.

Procedure

★ Q 4. Carry out the following procedure.

1. Place the 7486 IC on the breadboard.

2. Connect Pin 14 to +5 V.

3. Connect Pin 7 to 0 Volts or ground.

4. The first input A is applied to pin no 1. The input voltages can be taken directly from the power supply or the +5 V and 0 V lines on the breadboard.

5. The second input B is applied to pin no 2.

6. The output is obtained from pin no 3.

7. Double Click the xor.vi file.
8. In the **front panel** window click the **Run** button (shown by the arrow key).

9. Data acquisition starts, three waveforms starts appearing on the waveform graph.

10. The **legend bar** shows the different colors associated with signals.

   (a) White is the output signal.
   (b) Red is the first input signal on pin no. 1.
   (c) Green is second input signal on pin no. 2.

11. When the output is high, the LED on the front panel also turns green.

12. To stop acquiring signal, click the **Stop** button at the bottom of the waveform graph.

13. Now observe the outputs for all the logic combinations and verify the XOR truth table.

   ★ **Q 5.** Draw the Function Table on your note books also write down the voltages of the inputs and the output.

### 11.9 Filter Design

**Objective**

Having understood what is meant by frequency, the present section introduces the concept of a **filter**, its operation and design.

**Frequency content of signals**

A signal may be composed of one or more frequencies. Thus each signal could be expressed in terms of its frequencies as well.

**What is a filter?**

Filter is an electronic device which allows a range of certain frequencies to pass through and blocks the rest.
11.9. FILTER DESIGN

Cut-off or corner frequency

The cut-off frequency of the filter is the frequency at which the power output of
the filter is reduced to half of its maximum. This frequency is denoted by \( f_c \).

Figure 11.15 shows a filter made from a resistor \( R \) and a capacitor \( C \). This filter
is an RC filter. These components can be used for making both low pass as well
as high pass filters. We will describe these terms shortly. The input signal comes
into the left and the output is taken from the right.

![RC Filter Diagram](image)

**Figure 11.15:** An RC Filter.

Filter diagrams

![Filter Diagrams](image)

**Figure 11.16:** Filter Diagrams; the output power from the filter is plotted with
respect to frequency.

Figure 11.16(a) shows a low pass filter, (b) shows a high pass filter and (c) shows
a band pass filter. For example, a low pass filter allows low frequencies to pass
through and blocks high frequencies. The transition between the blocked and
the unblocked frequencies, however, is not sharp. The output power decreases
smoothly. At an applied frequency equal to \( f_c \), the output power is reduced to
half as compared to the maximum.
Design considerations

For RC circuits, cut-off frequencies are determined using the formula,

\[ f_c = \frac{1}{2\pi RC}, \]  
(A-2)

Calculational example

1. Select a suitable cut-off frequency for the filter.
2. Take any standard value of the resistor available.
3. Then, using Equation A-2, determine the value of capacitor, to achieve the desired \( f_c \).

Low pass filter

In order to make a low pass filter, we take the output across the capacitor, as shown in Figure 11.15. This will attenuate all the high frequencies present in the input signal and will allow the low frequency components to pass. Attenuation means that the filter will reduce the output amplitude for the high frequency components.

High pass filter

In order to make a high pass filter, we take the output across the resistor, as shown in Figure 11.17. This will attenuate all the low frequencies present in the input signal, allowing the high frequency components to pass.

![Figure 11.17: High Pass Filter.](image-url)
Mathematical explanation

As we know a resistor has a property called the resistance \( R \). The resistance quantifies the opposition to the flow of electrons. Similarly, a capacitor has an analogous property called the reactance denoted as \( X_c \). The capacitive reactance is

\[
X_c \propto \frac{1}{\omega C} = \frac{1}{2\pi f C}.
\]  

(A-3)

The above relation tells that capacitive reactance is inversely related to frequency and capacitance. Thus by decreasing frequency, reactance increases. When the frequency is zero, the reactance becomes infinite, indicating an open circuit, as shown in Figure 11.18(b). This means that no current will flow through the circuit and the voltage drop across the resistor \( R \) will be zero \((I_R = 0)\). But according to Kirchhoff’s rules, the output voltage must be equal to the input voltage. Hence the drop across the capacitor will be equal to the input voltage and hence, maximum. Likewise, for extremely high frequencies, the reactance of the capacitor drops to very small values and it behaves like a short circuit—an ordinary piece of wire. This is shown in Figure 11.18(c). The drop across the capacitor becomes zero and hence the output voltage is zero.

![Figure 11.18: (a) Behaviour of the low pass filter under the conditions of (b) DC input voltage and (c) very high frequencies.](image)

**Q 6.** Draw an illustration similar to Figure 11.18 for the high pass filter of Figure 11.17 and describe how this circuit attenuates low frequencies.

**Q 7.** When a DC voltage is applied to a capacitor, does the capacitor conduct current?

**Q 8.** When a sinusoid of any frequency is applied, the capacitor conducts current. However, in between the plates of the capacitor, we have air that is a poor conductor. How does the current pass through the insulator?
Procedure

Q 9. Carry out the following procedure.

1. Design a low pass filter with cut-off frequency of approximately 1.3 KHz using the provided components.
2. Place the components on the breadboard.
3. Generate a sine wave using the signal generator. This is going to be used as an input signal for the rest of our experiment.
4. Set the peak input voltage $V_{in}$ to 5 V.
5. The output can be viewed using LabVIEW, and the DAQ system. Connections will be shown by the demonstrator.
6. Run the filter.vi file.
7. In the front panel window click the Run button (shown by the arrow key).
8. Data acquisition starts, the filter’s output starts appearing on the waveform graph.
9. Observe the waveforms and carry out the following procedures.

Low Pass Filter

1. Take the output across the capacitor resulting in a low pass filter.
2. Now, you will successively change the input frequency, $f_{in}$ to 0.1, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 2.0, 3.0, 4.0, 5.0, 6.0, 7.0, 8.0, 9.0 and 10.0 KHz.
3. Complete Table 11.3 in your note books.
4. Use Matlab to plot the graph between $f$ and $V_{out}$.
5. Use Matlab to plot the graph between $f$ and $G$ that is called the gain of the filter. The Matlab command to compute the Gain is $G = 20 \cdot \log_{10}(V_{out}/V_{in})$
6. Select a suitable range of points in the data between $f$ and $G$. Fit these points to a straight line using the lsqcurvefit function in Matlab. What is the slope of this line? What does this slope represent?
### High Pass Filter

**Q 10.** For the high pass filter, carry out the following procedure.

1. Take the output across the resistor resulting implementing the high pass filter.

2. Successively change the input frequency, $f_{in}$ to 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0, 3.0, 4.0, 5.0, 10.0, 15.0 and 20.0 KHz.

3. Complete Table 11.4 in your note books.

4. Use Matlab to plot the graph between $f$ and $V_{out}$.

5. Use Matlab to plot the graph between $f$ and $G$.

### Table 11.4: Suggested format for tabulating the experimental results for the high pass filter.

<table>
<thead>
<tr>
<th>Frequency $f_{in}$ (KHz)</th>
<th>Peak output voltage $V_{out}$ (V)</th>
<th>$G = 20 \times \log \left(\frac{V_{out}}{V_{in}}\right)$ (dB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>:</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Q 11.** Based on the plotted results, describe your observations on the low and high pass filters.
11.10 Filtering a Composite Signal

Objective

This section introduces the use of an operational amplifier commonly abbreviated as the opamp. It also represents a more illuminating verification of the concept of filtering introduced in the previous Sections.

Composite Signal

Two or more signals are added to make what we call a composite signal. In this section we are using an operational amplifier to add the two signals.

A signal may possess different frequency components. When two signals are added, the resulting signal possess the frequency components of both signals. Thus in order to filter out a particular signal, we design a suitable corresponding filter (low pass, high pass, or band pass) to extract the desired components and leave the rest.

Block Diagram

The Block diagram shown in Figure 11.19, describes the flow chart of the experiment, in which two signals with different frequencies are added together using an opamp to produce a composite signal which is then filtered.

![Block diagram showing low pass filtering of the composite signal.](image-url)
Frequency Spectrum of Signals

★ Q12. Carry out the following procedure to see the frequency spectrum. Note down all the observations on your note books.

1. Generate a sine wave of 1 KHz.
2. Run the Labview file composite.vi.
3. In the front panel window click the Run button (shown by the arrow key).
4. See the sine wave on the waveform graph.
5. Start Matlab and type
   >> signalfft:
6. It shows the frequency spectrum of the signal. You can see a single peak on the frequency axis at 1 KHz.

★ Q13. Repeat the same procedure to see the frequency spectrum of a square wave of 1 KHz. It only consists of frequencies of 1KHz, 3KHz, 7KHz, and so on. It consists of odd harmonics of the fundamental frequency which is (1KHz in this case).

★ Q14. Carry out the following procedure to see the frequency spectrum.

1. Generate a sine wave of 1 KHz.
2. Generate another sine wave of 5KHz using the second signal generator.
3. The summer circuit has already been made for you on the breadboard. You give the input signals at proper places on the breadboard. These positions are marked on the breadboard.
4. The output composite signal can be viewed from pin no. 6 of the opamp on the breadboard.
5. Run the Labview file composite.vi and observe the frequency spectrum in Matlab.

★ Q15. Draw the composite signal waveform (both in time and frequency domain).

★ Q16. Draw the composite signal after low pass filtering (both in time and in the frequency domain).
Experiment

Q 17. Carry out the following procedural steps. The circuit diagram is shown in Figure 11.20.

1. The first input signal is taken from the signal generator. This input is a square wave of frequency 15 KHz.

2. Take another sine wave of frequency 1 KHz from the second signal generator.

3. The summer circuit has already been made for you on the breadboard. You give the input signals at proper places on the breadboard.

4. The output composite signal can be viewed from pin no. 6 of the opamp on the breadboard.

5. Run the Labview file composite.vi.

6. In the front panel window click the Run button (shown by the arrow key).

7. Data acquisition starts, the filter's output starts appearing on the waveform graph.

8. You can see the different frequencies in the composite signal.

9. Start Matlab and type

   >> signalspect;

10. Now make a low pass filter of cut off frequency approximately 1.3 KHz. Assemble it on the breadboard and filter the composite signal.

11. Observe the filtered output.

Q 18. Draw the composite signal waveform (both in time and frequency domain).

Q 19. Draw the composite signal after low pass filtering (both in time and frequency domain).

Q 20. Summarize your results and draw appropriate conclusions.
Figure 11.20: Circuit diagram for generation and low pass filtering of the composite signal.

11.11 Idea Experiments

1. How do we reduce noise? One method is to acquire more and more samples points, the signal grows more rapidly than the noise, increasing the signal-to-noise ratio. Design a simple experiment to verify this [4].

2. Determine the Boltzmann constant using measurements of Johnson noise [5].

3. The electrocardiogram (ECG) is an example of a scenario where the signal of interest can be buried in a high level of noise. Design a circuit that measures the ECG from a human body, enhancing the signal. Also filter out the interfering noise from the 50 Hz mains [6].
Bibliography


Chapter 12

Latent Heat of Vaporization of Liquid Nitrogen and Specific Heats of Metals

Waqas Mahmood, Sabieh Anwar and Wasif Zia

In this experiment, we have used a simple and intuitive setup to measure the latent heat of vaporization of liquid nitrogen and the specific heat capacity of a material. We will learn about the thermal properties of materials including solids, liquids and gases. Furthermore, we will be exposed to the safe handling of cryogens that are routinely used in low temperature physics. This experiment is inspired from previously published articles [1, 2] on the subject.

KEYWORDS Latent Heat of Vaporization · Specific Heat Capacity · Cryogenics · Wire wound resistors · Ammeter · Voltmeter · Cooling baths · Measurement of mass ·

APPROXIMATE PERFORMANCE TIME 4 hours

12.1 Conceptual Objectives

In this experiment, we will,
1. understand the concepts of latent heat of vaporization, internal energy, equipartition and specific heat,
2. understand the transfer of heat in calorimetric experiments,
3. determine the molar gas constant,
4. learn how to get meaningful data from experimental graphs,
5. learn calculations of uncertainties from experimental data,
6. practice error propagation, and
7. verify the Dulong and Petit’s Law.

12.2 Experimental Objectives

The experimental objectives attainable from this experiment are,
1. getting familiar with the safe use of cryogens,
2. making of solutions to achieve low temperature,
3. setup of simple circuits for heating and measurement of current and voltage,
4. handling of metals, and
5. using Matlab as a tool for data processing.

12.3 Equipartition of energy

We know that molecules are always on the move as they have kinetic energy, but the question is, how is this energy shared? James Clerk Maxwell solved this problem for a large number of molecules. He said that energy is equally divided in all the directions a molecule is free to move. The average energy, when the number of molecules is large, per molecule is $\frac{1}{2}k_B T$ for each independent degree of freedom, where $k_B$ is $1.38 \times 10^{-23}$ J/K, is known as the Boltzmann constant and $T$ is temperature in degrees kelvin. Each direction in which a particle is free to move is counted as a degree of freedom. For example if a particle can only move along the x axis, it has 1 degree of freedom. This principle of equal sharing of energy between the degrees of freedom is called the principle of equipartition of energy [3].
Let's model our description for nitrogen \((N_2)\) as we will be using its liquid form for our experiment. \(N_2\) is a diatomic gas and its total kinetic energy is the sum of translational kinetic energy \(E_T\) and rotational kinetic energy \(E_R\). (We ignore the vibrational degrees of freedom as these vibrations only occur at very high temperatures, much higher than those achievable in the present experiment.) The total energy inside all the molecules in the system is often referred to as the internal energy.

Translational kinetic energy results in a net displacement of the center of mass in any direction as explained by Figure 12.1. Mathematically,

\[
E_T = \frac{1}{2}m
\nu_x^2 + \frac{1}{2}m\nu_y^2 + \frac{1}{2}m\nu_z^2.
\]  
(A-1)

![Diagram of a diatomic molecule showing three axes and two spirals representing the directions of rotation](image)

Figure 12.1: The diagram is a representative sketch of a diatomic molecule, showing the three directions in which a diatomic molecule may displace and the spirals on \(x\) axis and \(y\) axis show the two possible directions of rotation. This accounts to a total of five degrees of freedom.

where \(m\) is the mass of the molecule and \(\nu_x, \nu_y, \nu_z\) are the velocities in relevant directions. Further, in Figure 12.1 there are two spirals showing the possible directions of rotation about the center of mass. These are the rotational degrees of freedom and result in the rotational kinetic energy,

\[
E_R = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2.
\]  
(A-2)

where \(I_x\) and \(I_y\) represent the moments of inertia along that axis and \(\omega_x\) and \(\omega_z\) are the respective angular velocities. Counting the number of terms in both Equation A-1 and Equation A-2 we have a total of five degrees of freedom. Now, using the equipartition theorem we may say that the internal energy \((E = \)
$E_T + E_R$ for $N$ molecules is,

$$E = N \frac{5}{2} k_B T. \quad (A-3)$$

**Q 1.** What is the mathematical expression of internal energy for a monatomic gas such as He?

**Molar heat capacity of solids**

Solids have atoms vibrating in fixed positions. At these specific positions they have kinetic energy which allows them three degrees of freedom, furthermore they have potential energy in all three directions. This means that the total internal energy for molecules in a solid is due to six degrees of freedom. Therefore using the equipartition theorem,

$$E = N3k_B T. \quad (A-4)$$

The heat capacity $C$ signifies that a small change in temperature $\Delta T$ leads to a change in total internal energy $\Delta E$,

$$C = \frac{\Delta E}{\Delta T}. \quad (A-5)$$

Using Equation A-5 we get,

$$C = 3N k_B. \quad (A-6)$$
As the number of molecules is given by the product of number of moles \( n \) and the number of molecules per mole, the Avogadro number, \( N_A \),

\[
C = 3nN_A k_B. \tag{A-7}
\]

The heat capacity for one mole of the solid is called the \textit{molar heat capacity} \( C_M \) and is given by,

\[
C_M = 3N_A k_B = 3R. \tag{A-8}
\]

where \( R = N_A k_B = 8.31 \) J/mol K, is the molar gas constant.

It is important to note that according to Equation A-8, \( C_M \) is a \textit{constant}, which means that the molar heat capacity is independent of temperature and the type of solid: all solids should have the same molar heat capacity. This is known as the \textit{Dulong–Petit} law [3].

\textbf{Limitations to the classical equipartition theorem}

Experimentalists found that Dulong–Petit law was only obeyed for temperatures above 300 K. However, for temperatures below 250 K a heavy dependence of heat capacity on temperature was observed. A general sketch of this dependence is shown in Figure 12.3.

![Graph showing the variation of \( C_M \) with temperature](image)

Figure 12.3: The experimentally observed variation of \( C_M \) with temperature, showing the region where Dulong–Petit law agrees with experimental results and also the region where the law breaks down.

The heat capacity of solids is important information as it allows us to understand
the material in a myriad of ways. Therefore an accurate depiction is necessary. Further, it was important for physicists to explain the theory behind the observation.

At one time in history, physicists considered an atom to be indivisible. The advent of quantum mechanics led scientists to understand that the atom is not indivisible and that the thermal energy is divided between the ions and electrons. Thus there is a fraction of thermal energy carried by electrons and a fraction carried by the ions. Both of these factors contribute to the specific heat. The mathematical explanation of the low temperature dependence of heat capacity is called the Einstein-Debye model. Put very simply, the electrons and ions inside the crystal structure of the solid are oscillating. The quantum theory predicts that these oscillations are quantized, they occur only in discrete steps. This discretization becomes more prominent at lower temperatures, resulting in departure from the Dulong-Petit law.

**Latent Heat of Vaporization**

The amount of energy released or absorbed by any substance during a phase transition is called the latent heat. If we add heat continuously as in Figure 12.4, a change of phase from solid to liquid and then from liquid to vapor occurs. These changes are called phase transitions. The latent heat absorbed during the liquid-vapor transition is called the latent heat of vaporization. This energy overcomes the inter-molecular forces inside the liquid. Figure 12.4 illustrates this phenomenon, whereby temperature remains constant as the heat is supplied at the phase transition.

The latent heat of vaporization can be mathematically expressed as,

$$L_v = \frac{\Delta Q}{m}. \quad (A-9)$$

where \(\Delta Q\) is the heat supplied during phase transition and \(m\) is the mass of the liquid vaporized.

In our experiment we will use electrical energy to supply energy. Current is made to flow through a heater placed inside the liquid of interest, in our case, liquid nitrogen. The heat supplied is, \(\Delta Q = VI\Delta t\), where \(V\) is the voltage from the source, \(I\) is the current flowing and \(\Delta t\) is time interval for which electrical heating remains on.

How do we calculate the mass of the nitrogen vaporized due to the electrical
Figure 12.4: Phase change when heat is added at a constant rate. The temperature remains constant during the phase transition and the heat supplied at these points is called a latent heat.

heating alone? In fact, the mass of liquid nitrogen, gradually decreases because the room temperature provides a sufficiently high temperature for the nitrogen to boil off. So, one has to separate out the loss in mass due to electrical heating from the loss in mass from ambient heating. The term \( m \) in Equation A-9 corresponds to the decrease in mass due to electrical heating only. This is done by establishing a background rate of loss of nitrogen while electrical heating is switched off. In this experiment, you will be required to intelligently interpret your data and calculate \( m \) by comparing against the background loss.

With electrical heating, the preceding equation becomes,

\[
L_v = \frac{V \Delta t}{m}.
\]

12.4 Experimental preparation and safety measures

Using Liquid Nitrogen

Liquid nitrogen is a colorless, odorless and tasteless fluid which boils at 77 K, and is formed by cooling and increasing pressure on air which is predominantly \( \text{N}_2 \). On evaporation, it generates enormous pressure and direct contact with liquid nitrogen can cause cold burns or frost bites. Liquid nitrogen should never be mixed with water and you must wear goggles when making solutions.
Cooling baths

We will make few low temperature cooling baths in this experiment using different solvents. In the process of making these baths, fumes are produced. To avoid fumes entering the eyes, safety goggles are used. These baths have low temperature and direct contact might cause injury. You must wear the provided gloves as a safety precaution. Some cooling baths along with the temperatures achievable are given in Table 12.1.

<table>
<thead>
<tr>
<th>S.No</th>
<th>Mixture</th>
<th>Temperature °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Air</td>
<td>29</td>
</tr>
<tr>
<td>2</td>
<td>CaCl₂·6 H₂O/ice 1 : 2.5</td>
<td>-10</td>
</tr>
<tr>
<td>3</td>
<td>NaCl/ice 1 : 3</td>
<td>-20</td>
</tr>
<tr>
<td>4</td>
<td>Ethanol/N₂</td>
<td>-40</td>
</tr>
<tr>
<td>5</td>
<td>Acetone/CO₂</td>
<td>-78</td>
</tr>
<tr>
<td>6</td>
<td>Liquid N₂</td>
<td>-196</td>
</tr>
</tbody>
</table>

Table 12.1: Mixtures with different approximate temperatures.

Some solvents when mixed with dry ice are flammable but most of them are not. Students must not mix any solvent without prior knowledge.

Handling of the Dewar

The container of liquid nitrogen also called a dewar as shown in Figure 12.5, should be handled with care and covered properly after taking out liquid nitrogen. The cylindrical tubes used to take out nitrogen are delicate and no extra pressure should be exerted on them. The container must be refilled when the level of LN₂ is below a certain value.

Q 2. How can the container be damaged if liquid nitrogen is not refilled at the right time?

Using Metals

We will use lead and tin in our experiment. Lead is carcinogenic (cancer causing) and tin is toxic, therefore it is advised that students handle this material with tongs, wear the provided gloves and try not to make direct contact with hands.
In case of contact, wash your hands properly with water.

12.5 The Experiment

A resistor (e.g., a 10 watts wire wound resistor shown in Figure 12.6 or a disassembled 30 watts soldering rod) is connected to the variac (Space Power Electronics, Karachi) through a needle-type ammeter in series as shown in the Figure 12.7. The variac is an ac transformer that can provide variable voltage. The voltage is measured by the voltmeter.

![Wire wound resistor](image)

Figure 12.6: Wire wound resistor.

**Q 3.** Liquid nitrogen from the cryogenic container is poured safely into the provided styrofoam cup and the cup is placed on the weighing balance which will record the loss in the mass of liquid nitrogen. This decrease in the mass of the liquid nitrogen is recorded against time and is measured with the stop watch.

**Q 4.** Why does the mass of the liquid nitrogen decrease and at what rate? Can this rate be controlled?

---

1. Wire wound resistors consist of a cylindrical core which is wrapped with a wire. This core is typically made up of a ceramic material and the wire is a type of resistance wire. Wire wound resistors are a type of power resistors and are very accurate.
Figure 12.7: Set up of the Experiment.

★ Q 5. After having the background loss for a few minutes, the switch is closed to turn on the heater. Now the rate of mass loss will be faster and is again recorded as a function of time. The heater is then switched off to reestablish the background loss rate.

★ Q 6. Plot the data points using Matlab. Find the overall change in mass during the process of heating.

★ Q 7. From the data, find the change in mass \( m \) only due to electrical heating.

★ Q 8. Calculate the latent heat of vaporization of liquid nitrogen \( L_v \).

★ Q 9. What is the uncertainty in your calculated value of \( L_v \) for liquid nitrogen?

**Specific Heat Capacity Measurement**

★ Q 10. Follow these steps to calculate the specific heat capacity of the metal (tin or lead).

1. Fill the styrofoam cup with liquid nitrogen safely and place it on the weighing balance.
2. Measure the mass of the cube of lead or tin.

3. Place the cube of lead beside the styrofoam cup on the balance.

4. Measure the background loss for a few minutes, with the cube placed on the balance.

5. Place the lead cube inside the cup for a certain duration of time and record the mass evaporated.

6. Remove the solid from the cup and measure the background loss again for a few minutes.

7. Plot the results, extrapolate and intelligently using the data, calculate the specific heat capacity using the relation.

★ Q11. Explain, preferably with a mathematical formula, your derivation for the specific heat capacity.

★ Q12. The specific heat capacity is a temperature dependent quantity. Describe, at what temperature is your calculated value good for.

★ Q13. Estimate the value of molar gas constant \( R \).

★ Q14. Calculate uncertainties in the values of \( C \) and \( R \).

Low Temperature Specific Heats

In order to verify Dulong and Petit law as well as its deviations, we have to calculate the values of specific heat at different temperatures and plot a graph between temperature and specific heat values. Due to time constraints, we can only do this with the metal initially placed at room temperature and maybe, for a couple of times, when the metal is placed in some cooling bath.

★ Q15. For the low temperature measurement, follow these steps.

1. Make a cooling bath and measure its temperature with the help of the provided thermocouple.

2. Measure the background loss of liquid nitrogen for a few minutes.

3. Insert the material from the cooling bath to the liquid nitrogen cup and measure the decrease in mass with time.
4. When the sizzling sound stops, equilibrium is reached and background is reestablished.

5. From the data acquired, find the specific heat capacity for the low temperature.

6. Repeat for different initial cooling baths.
Bibliography


Chapter 13

Electromagnetic Induction
and Read-Write Operations in
Magnetic Media

Umer Hassan, Wasif Zia and Sabieh Anwar

Why does a magnet rotate a current carrying loop placed close to it? Why does the secondary winding of a transformer carry a current when it is not connected to a voltage source? How does a bicycle dynamo work? How does the Mangla Power House generate electricity? Let’s find out the answers to some of these questions with a simple experiment.

KEYWORDS

Faraday’s Law · Magnetic Field · Magnetic Flux · Induced EMF · Magnetic Dipole Moment · Hall Sensor · Solenoid

APPROXIMATE PERFORMANCE TIME 4 hours

13.1 Conceptual Objectives

In this experiment, we will,

1. understand one of the fundamental laws of electromagnetism,
2. understand the meaning of magnetic fields, flux, solenoids, magnets and electromagnetic induction,

3. appreciate the working of magnetic data storage, such as in hard disks, and

4. interpret the physical meaning of differentiation and integration.

### 13.2 Experimental Objectives

The experimental objective is to use a Hall sensor and to find the field and magnetization of a magnet. We will also gain practical knowledge of,

1. magnetic field transducers,

2. hard disk operation and data storage,

3. visually and analytically determining the relationship between induced EMF and magnetic flux, and

4. indirect measurement of the speed of a motor.

### 13.3 The Magnetic Field $B$ and Flux $\Phi$

The magnetic field exists when we have moving electric charges.

About 150 years ago, physicists found that, unlike the electric field, which is present even when the charge is not moving, the magnetic field is produced only when the charge moves. This discovery allowed physicists to learn interesting ideas about materials. In the twentieth century, scientists determined the configuration of elementary particles in atoms and they realized that electrons inside atoms also produce tiny magnetic fields. This field is found in all materials. The magnetic field is mapped out by magnetic field lines.

Magnetic field lines are like stretched rubber bands, closely packed near the poles. This is why the closer we get to the poles of a magnet, the higher the magnetic field. The number of magnetic field lines passing through an area is known as magnetic flux $\Phi$.

Mathematically, we divide an area through which we want to find the flux into identical area elements $\Delta \mathbf{A}$ perpendicular and away from the surface as shown in
the Figure 13.1. A scalar product between the magnetic field vector $\vec{B}$ and $\Delta \vec{A}$ is,

$$\phi = \vec{B}_1 \cdot \Delta \vec{A}_1 + \vec{B}_2 \cdot \Delta \vec{A}_2 \ldots \quad (A-1)$$

Subsequently, we may also write

$$\phi = \sum \vec{B}_i \cdot \Delta \vec{A}_i \quad (A-2)$$

![Figure 13.1: Magnetic flux through an area.](image)

### 13.4 Electromagnetic Induction

Extensive work was done on current carrying conductors in the nineteenth century. Major ground work was set by Faraday (1831) and following him Lenz (1834) [1]. Faraday discovered that a changing magnetic field across a conductor generates an electric field. When a charge moves around a closed circuit this electric field does work on the charge. Like the electromotive force (EMF) of a battery this induced EMF is capable of driving a current around the circuit.

Faraday’s law asserts that the EMF produced is directly proportional to the rate at which the magnetic field lines per unit area or magnetic flux ‘cuts’ the conducting loop. Lenz’s law is incorporated into Faraday’s Law with a negative sign which shows that the EMF produced opposes the relative motion between the conductor and magnet, it tries to resist the change in flux.

Mathematically both of these laws are expressed together as,

$$\varepsilon = -\frac{d\phi}{dt} \quad (A-3)$$

for a single loop of conductor, where $\varepsilon$ is the electromotive force induced, $\phi$ is the magnetic flux. $\frac{d\phi}{dt}$ is time rate of change of magnetic flux. The rate depends
on the speed at which the magnet moves relative to the conductor loop, as well as the strength of the field.

Electric power plants or more commonly, generators, are a physical manifestation of laws of induction. The principle is to change the magnetic flux over large stationary coils. The 'change' of flux is brought about mechanically, either by falling water or by running a turbine. The changing flux induces an EMF in the coils.

**Q 1.** What are the units of $\varepsilon$ and $\phi$?

**Q 2.** Rewrite Equation 13.2 for $N$ number of loops. How does the EMF depend on $N$?

### 13.5 Solenoids

Shown in Figure 13.3 is a coil of wire wound around a core. Magnetically it behaves like a bar magnet, producing a magnetic field when the current flows. It remains a magnet till the time current is flowing through the conductor.

The mathematical expression for magnetic field generated inside an *ideal solenoid* is,

$$B = \mu_0 n I,$$

(A-4)

where $\mu_0$ is the permeability in free space, value: $1.26 \times 10^{-6}$ H/m, $n$ is the number of turns of the conductor per unit length and $I$ is the current through the conductor. The magnetic field $B$ is measured in Tesla (T) or Gauss (G), where
1 G equals $10^{-4}$ T. In our experiment we will use a changing magnetic field near

![Solenoid diagram](image)

Figure 13.3: Solenoid made from an enameled copper wire wound on a plastic pipe

a solenoid to induce an EMF in it. This is the the Faraday effect!

## 13.6 The Hall effect

Imagine a sea. There is a sea of electrons in a conductor. When we apply a potential this 'sea' flows from the higher to the lower potential. Further, if we place this conductor, in which current is flowing in a magnetic field the moving charges tend to interact with the applied magnetic field and also deflect. This deflection results in a potential difference across or perpendicular to the conduction path, known as the Hall voltage.

Figure 13.4 illustrates how moving charges are deflected due to the applied magnetic field. The magnitude of this force ($F_B$) is given by,

$$F_B = Bqv.$$  \hspace{1cm} \text{(A-5)}

where $q$ is the charge and $v$ is the velocity. The build-up of charges on one side generates an electric field ($E_\perp$) perpendicular to the current as shown in Figure 13.4. These charges continue to accumulate till the time force ($F_E$) due to electric field,

$$F_E = qE_\perp.$$  \hspace{1cm} \text{(A-6)}

is equal to the force due to the magnetic field ($F_B$). Mathematically this equilibrium means that,

$$F_E = F_B.$$  \hspace{1cm} \text{(A-7)}

or

$$Bqv = qE_\perp.$$  \hspace{1cm} \text{(A-8)}
The voltage developed due to $E_\perp$ is,

$$V_H = E_\perp w. \quad (A-9)$$

where ($V_H$) is the Hall voltage and $w$ is the width of the conductor. Combining Equation A-8 and Equation A-9 we get,

$$V_H = uwB. \quad (A-10)$$

We know that the average velocity of electrons in terms of current ($I$) is given by,

$$\nu = \frac{I}{neA}. \quad (A-11)$$

where $n$ is the volume density of electrons and $A$ is the cross-sectional area, a product of width ($w$) and thickness ($T$).

Combining Equation A-11 and Equation A-10 we obtain the Hall voltage in terms of applied magnetic field,

$$V_H = \frac{BI}{neT}. \quad (A-12)$$

The Hall effect is important in the study of materials, for example it helps us to find the number of conducting particles in a wire and their charge. In our experiment, this effect holds a central importance as we will use sensors developed using this principle to probe the magnetic fields generated by magnets. Read heads in tape recorders and magnetic disk drives utilize this principle too.
**Q 3.** A strip of copper 150 μm thick is placed inside a magnetic field \( B = 0.65 \, \text{T} \) perpendicular to the plane of the strip, and a current \( I = 23 \, \text{A} \) is setup in the strip. What Hall potential difference would appear across the width of the strip if there were \( 8.49 \times 10^{28} \) electrons/m²?

**Comparison between the solenoid and the Hall probe**

The Hall probe and the solenoid are both transducers, they convert one form of energy to another. Figure 13.5 shows that Hall probes generate a measurable potential which varies with the direction and magnitude of the field and flux. This potential is then converted to magnetic field using a simple relation provided by the manufacturer of the Hall chip.

On the other hand, a solenoid, directly measures the EMF. The value of EMF, of course depends on the rate of change of flux being measured. However a major role is also played by the number of turns of the solenoid.

![Diagram](image)

Figure 13.5: A comparison between the operation of the solenoid and Hall probe.

**13.7 Data Storage on a hard disk**

Computers are digital. So, every letter of every language must be stored or processed in computers in “digital form”; i.e. as a sequence of 0’s and 1’s. Computers use ASCII (American Standard Code for Information Interchange). It is a 7 bit code for all English alphabets, Roman letters and many other symbols. We will be using a similar scheme for a 5 bit code in our experiment. Consider the
following table, Table 13.1, which shows a possible binary conversion of English
alphabets into bits.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>00000</td>
<td>j</td>
<td>01001</td>
<td>s</td>
<td>10010</td>
</tr>
<tr>
<td>b</td>
<td>00001</td>
<td>k</td>
<td>01010</td>
<td>t</td>
<td>10011</td>
</tr>
<tr>
<td>c</td>
<td>00010</td>
<td>l</td>
<td>01011</td>
<td>u</td>
<td>10100</td>
</tr>
<tr>
<td>d</td>
<td>00011</td>
<td>m</td>
<td>01100</td>
<td>v</td>
<td>10101</td>
</tr>
<tr>
<td>e</td>
<td>00100</td>
<td>n</td>
<td>01101</td>
<td>w</td>
<td>10110</td>
</tr>
<tr>
<td>f</td>
<td>00101</td>
<td>o</td>
<td>01110</td>
<td>x</td>
<td>10111</td>
</tr>
<tr>
<td>g</td>
<td>00110</td>
<td>p</td>
<td>01111</td>
<td>y</td>
<td>11000</td>
</tr>
<tr>
<td>h</td>
<td>00111</td>
<td>q</td>
<td>10000</td>
<td>z</td>
<td>11001</td>
</tr>
<tr>
<td>i</td>
<td>01000</td>
<td>r</td>
<td>10001</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 13.1: Binary Representation of English alphabets

13.8 The Experiment

Building a Hall probe

Hall probes are used to measure magnetic fields. The output voltage of a Hall
sensor is proportional to the magnetic field being measured. The measured volt-
age is then converted to magnetic field using a calibration scheme provided by
the manufacturer of the Hall sensing chip. This calibration curve will be given to
you in the lab.

Hall probe sensor is shown in Figure 13.6. \(V_{cc}\) is provided to the Hall sensor using
the Universal Serial Bus (USB) port. All USB ports have a 5 V regulated output,
so we will be using USB port as power supply to the Hall chip.

![Figure 13.6: Hall sensor](image)
Magnetic Field of a Disk Magnet

Now let’s map the field of a disk magnet using the probe you just built. The lab has provided you small disk magnets based on iron.

![Diagram of a disk magnet with a Hall probe](image)

Figure 13.7: Schematic shows disk magnet field mapping using a Hall probe.

**Q 4.** Following the scheme in Table 13.2 below to find the output voltage on the probe as you move along the magnetic axis as shown in Figure 13.7. Make sure that the flat face of the probe is perpendicular to the magnetic axis.

<table>
<thead>
<tr>
<th>Distance (mm)</th>
<th>Output Voltage (volts)</th>
<th>Measured magnetic field $B_{\text{measured}}$ (Gauss)</th>
<th>$B_{\text{measured}}$ (Tesla)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>...</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 13.2: Mapping the field of a disk magnet. For voltage to field conversion use the provided calibration sheet.

**Q 5.** Find the error $B_{\text{measured}}$.

**Q 6.** Plot a graph between magnetic field strength $B_{\text{measured}}$ and distance.

**Q 7.** Using the above graph, write your observations regarding the change in magnetic field with respect to distance.
Magnetization of a disk magnet

Magnetic materials are made up of atoms which have magnetic dipole moment \( \vec{\mu}_B \). These randomly aligned dipoles have a net magnetic dipole \( \vec{\mu} \) if we sum over a volume \( V \), mathematically,

\[
\vec{\mu} = \sum \vec{\mu}_B. \tag{A-13}
\]

We can now define magnetization \( \vec{M} \) as

\[
\vec{M} = \sum \frac{\vec{\mu}_B}{V}. \tag{A-14}
\]

For a disk magnet the expression for the magnetic field strength as a function of distance is

\[
B(x) = \frac{\mu_0 M}{2} \left( \frac{x + l/2}{\sqrt{(x + l/2)^2 + a^2}} - \frac{x - l/2}{\sqrt{(x - l/2)^2 + a^2}} \right). \tag{A-15}
\]

where \( M \) is the magnetization of a disk magnet, \( x \) is the distance along the magnetic axis from the disk magnet, \( l \) is the thickness of the disk magnet and \( a \) is the radius [4].

The term in brackets needs some mathematical detail in which we will not delve.
However it is important to tell that it is a geometrical term which is the result of an integral depending upon the dimensions of the magnet and solved over the distance at which we are measuring the field.

For the sake of simplicity let's replace

\[
\frac{\mu_0}{2} \left( \frac{x + l/2}{\sqrt{(x + l/2)^2 + a^2}} - \frac{x - l/2}{\sqrt{(x - l/2)^2 + a^2}} \right) \tag{A-16}
\]

with the geometrical function \( f(x) \), obtaining

\[
B(x) = M f(x). \tag{A-17}
\]

The goal is to find the magnetization of the disk magnet using Equation A-17.

**Q 8.** For this perform the following procedure.

1. Find the thickness of the disk magnet with vernier callipers.
2. Find the radius of the disk magnet.
3. Run Matlab.
4. Enter values of \( d \), then type \( >> \text{magneticfield}(d) \);
5. The programme prompts to enter radius and thickness where \( d \) is the distance from the magnet at which magnetic field was measured.
6. The Matlab code returns the value of $f(x)$ that should be equal to $\frac{B(x)}{M}$.

7. Using the values of $B_{\text{measured}}$ and $f(x)$, evaluate the value of $M$ for each distance using Equation A-17.

★ Q 9. How has the error propagated in $M$? Find the mean and error in $M$?

★ Q 10. What are the units of $M$?

★ Q 11. Curve fit $f(x)$ and $B_{\text{measured}}$ to a suitable fitting function using least square curve fitting technique and evaluate the value of $M$.

13.9 Hard disk operation

Now we are going to simulate the operation of a hard disk. The experimental setup consists of a AC motor which rotates a disk. There is also a switch and a regulator, to control the speed, as shown in the schematic Figure 14.10. Magnets are placed over the disk at fixed positions.

There are two coils in an AC motor; a main winding and an auxiliary winding. The capacitor inside the box is connected in series with the stator winding and in parallel with the rotor winding, there is a difference in phase provided when a AC current passes through the capacitor, this difference in phase allows a couple of force to act on the axle which makes it rotate.

![Image of a motor showing its components: Soft Iron core, Bushing assembly, Coil, Axle]

Figure 13.8: Shows dismantled motor of a fan. Main and auxiliary windings are not visible as they are taped together.

There are two possible ways of placing a magnet, i.e., either place its north or south facing upward. When the north is upward we call it 1 and when south is upward we call it 0. As the disk rotates the magnetic flux linking the solenoid
with the Hall probe is changed with time. In order to see the induced EMF and magnetic flux we use the solenoid and the Hall probe respectively in different experiments.

![Diagram of hard disk experiment]

**Figure 13.9:** Setup of the hard disk experiment.

**Observing induced EMF and changing magnetic flux using a solenoid**

![Graph of EMF vs. time]

**Figure 13.10:** EMF induced in the solenoid as the magnet passes below the solenoid.
Figure 13.11: Magnetic North as measured by the Hall probe.

★ Q 12. Carry out the following procedure.

1. Make a solenoid approximately 8 cm in length. Note the number of turns. You have been provided with enameled copper wire.

2. Align the solenoid with the magnetic field of the magnets.

3. Turn ON the power supply, the motor starts and disk starts rotating.


5. Click on the Run button, the data starts acquiring.

6. Now, observe the waveform graphs.

7. Observe the EMF on the graph labeled as EMF.

8. Observe the magnetic flux on the graph labeled as Magnetic flux.

★ Q 13. You will observe something similar to Figure 13.10. Explain the graphs you observed. Is there a mathematical relation between the graphs you see? (HINT: Use the concepts of differentiation and integration.)

★ Q 14. Now change the number of turns of solenoid and observe the induced EMF. Draw the figures and note down your observations in your note books.

★ Q 15. What are your major conclusions?

Observing magnetic field using Hall Probe

Now instead of the solenoid we'll use the Hall probe as a magnetic field transducer. The voltage across the Hall chip is proportional to the magnetic flux. Depending
on the front (flat) or the back (round) surface or the direction of the field, voltage will either drop below or jump above the voltage when there is no field present. Figure 13.11 shows the corresponding flux from a typical experiment, magnet facing North.

★ Q 16. Now carry out the following procedure to observe the behavior of the Hall Probe.

1. Place the Hall probe sensor close enough the magnets.

2. Turn ON the supply, the motor starts and disk starts rotating.

3. Now Run the "HallProbe.vi" file.

4. Click on the Run button, the data starts acquiring.

5. Now, observe the waveform graphs.

6. Observe the Hall voltage on the graph labeled as voltage.

★ Q 17. Identify the south of the magnet.

★ Q 18. Note down your observations and inferences in your note books.

★ Q 19. What’s the difference you observe when using the Hall probe sensor from the solenoid? Write it down in your note book.

★ Q 20. How can we find the EMF from your Hall voltage observations? Sketch the curve of EMF.

★ Q 21. Can you come up with a method to measure the speed of the motor? Describe.

Data Reading and Writing Operation

This section illustrates how the data is read from a hard disk.

1. Place the magnets on the disk in some orientation, at the indicated positions. Such that it forms a letter. Your task is to identify what that letter is?

2. Align the Hall probe with the magnetic field of the magnets.
3. Turn ON the supply, the motor starts and disk starts rotating.


5. Click on the Run button, the data starts acquiring.

6. Now, observe the waveform graphs.

7. Observe the Hall voltage and the magnetic field.

8. Run Matlab and load letterRead.m file.

9. The stored letter is displayed.

10. Compare your result with the Table 13.1.

★ Q 22. Why do we need a starter sequence magnet?

Q 23. Change the orientation of the magnets and display a letter of your choice.
Bibliography


Chapter 14

Vibrations on a String and Resonance

Umer Hassan and Muhammad Sabieh Anwar

How does our radio tune into different channels? Can a music maestro shatter a crystal glass by beating the tabla with a particular frequency and pitch? How does our ear distinguish between tones in the multitude of sounds we hear every day? The answer to all of these questions lies in understanding the concept of resonance. The idea was discovered by Galileo Galilei with his investigations of pendulums beginning in 1602. The present experiment gives you an introduction to the phenomenon of resonance, and the frequencies at which it occurs by visualizing the stationary waves formed on a vibrating string. We hope you will enjoy this exercise of exciting and detecting standing waves on a string.

KEYWORDS

Transverse wave · Longitudinal wave · Wave interference · Resonance · stationary waves · Circular modes · normal modes

APPROXIMATE PERFORMANCE TIME 4 hours.

14.1 Conceptual Objectives

In this experiment we will learn,
1. the concept of wave,
2. the difference between transverse and longitudinal waves,
3. the phenomenon of wave interference,
4. the concept of wave vector and wave number,
5. derivation of the wave speed, and
6. the formation of stationary waves.

14.2 Experimental Objectives

The experimental objectives for the experiment include,

1. exciting and detecting standing waves on a string,
2. being able to identify where resonance occurs,
3. distinguishing linear from nonlinear behaviors, and
4. correlating experimental plots with mathematical relationships.

14.3 Waves and their different types

What is a wave?

A wave is a disturbance or variation that transfers energy progressively from point to point in a medium. It may take, for example, the form of an elastic deformation or a variation of pressure, electric intensity, magnetic intensity, electric potential, or temperature. A medium is a substance or material which carries the wave. The medium through which wave propagates may experience some local oscillations as the wave passes but the particles of the medium don’t travel along with the wave. Remember that waves involve the transport of energy without the transport of matter. However, we all know that waves can also travel through vacuum.

Mechanical Waves

A mechanical wave is a wave which is not capable of transmitting its energy through vacuum. Mechanical waves require a medium in order to transport their
energy from one location to another. Sound waves, water waves, and rope waves are examples of mechanical waves. Mechanical waves can be categorized into the following two main groups.

1. **Longitudinal Waves**
   A longitudinal wave is a wave in which particles of the medium move in a direction parallel to the direction which the wave moves. For example, when a spring under tension is set oscillating back and forth at one end, a longitudinal wave travels along the spring. It is composed of compressions and rarefactions, and is shown in Figure 14.1.

   ![Figure 14.1: Longitudinal wave in a helical spring.](image)

2. **Transverse Waves**
   A transverse wave is a wave in which particles of the medium move in a direction perpendicular to the direction which the wave moves. For example, when a string under tension is set oscillating back and forth at one end, a transverse wave travels along the string; the disturbance moves along the string but the string particles vibrate at right angles to the direction of propagation of the disturbances, as shown in Figure 14.2. The waves on a vibrating string are the type of transverse waves. These are the waves we will encounter in the present experiment. We may recall that light or electromagnetic waves are also transverse waves.

   ![Figure 14.2: Illustration of particle motion w.r.t to wave propagation in a transverse wave.](image)
14.4 Wave Interference and Resonance

Interference of waves

Interference is the phenomenon occurring when two waves meet while traveling along the same medium. Constructive interference is a type of interference which occurs at any location along the medium where the two interfering waves have a displacement in the same direction. In other words, when the crest or trough of one wave passes through, or is super positioned upon, the crest or trough respectively of another wave, the waves constructively interfere. When waves interfere, amplitudes add. Figure 14.4 shows the two waves of different amplitudes constructively interfering to give a resulting wave of increased amplitude.

Destructive interference is the type of interference which occurs at any location along the medium where the two interfering waves have a displacement in the opposite direction. When the crest of one wave passes through, or is super positioned upon, the trough of another wave, we say that the waves destructively...
Figure 14.3: Illustration of variation of amplitude, frequency and phase of harmonic waves, (a) amplitude of one wave is twice the other, (b) frequency of one wave is twice the other, and (c) one wave has a phase difference of $\frac{\pi}{2}$ with the other.

Interference. We often say that when waves interfere, amplitudes add. During destructive interference, since the positive amplitudes from one crest are added to the negative amplitudes from the other trough, this addition can look like a subtraction. Refer to Figure 14.5 for a demonstration of this concept.

If the phase difference is close to 180°, the resultant amplitude is nearly zero. When $\phi$ is exactly 180°, the crest of one wave falls exactly on the valley of
Figure 14.4: Constructive wave interference phenomenon. (a) a wave of 5 units amplitude, (b) a wave of 7 units amplitude, and (c) resulting wave of 12 units amplitude. The resultant amplitude is zero, corresponding to total destructive interference.

Figure 14.5: Destructive wave interference phenomenon. (a) a wave of 5 units amplitude, (b) a wave of -4 units amplitude, and (c) resulting wave of 1 unit amplitude.

★ Q1. Two waves travel in the same direction and interfere. Both have the same wavelength, wave speed and an amplitude of 10 mm. There is a phase difference of 110° between them. (a) What is the resulting amplitude due to wave interference? (b) How much should the phase difference change so that the resultant wave has an amplitude of 5 mm?
Standing or stationary waves

Standing waves are formed by the interference of two harmonic waves of the same amplitude and frequency (and therefore same wavelength), but traveling in opposite directions. Due to the interference of the two waves, there are certain points called nodes at which the total wave is zero at all times. The distance between two consecutive nodes is exactly half the wavelength. The points at the middle between consecutive nodes are called anti-nodes. At the anti-nodes the total wave oscillates with maximum amplitude, equal to twice the amplitude of each wave. Anti-nodes are also half a wavelength apart as shown in Figure 14.6.

![Standing wave](image)

Figure 14.6: Standing wave.

Behavior at the boundary

The behavior of a wave upon reaching the end of a medium is referred to as boundary behavior. We consider two kinds of behaviors at boundaries.

Reflection from a free end

If the boundary is not stationary and is vibrating or moving, it is called as a free boundary. If a wave travels towards the free boundary, the last particle of the wave can no longer interact with the first particle of the free end. Since the rope and boundary are no longer attached and interconnected, they will slide past each other. So when a crest reaches the free end, the last particle of the wave receives the same upward displacement; only now there is no adjoining particle to pull downward upon the last particle of the wave to cause it to be inverted. The result is that the reflected wave is not inverted. When an upward displaced wave
is incident upon a free end, it returns as an upward displaced wave after reflection and vice versa. Inversion is not observed in free end reflection, as shown in Figure 14.7.

![Wave Diagram](image)

Figure 14.7: Reflected wave from free end doesn’t get inverted.

**Reflection from a fixed end**

If a boundary is stationary i.e. not vibrating but fixed and a wave is traveling towards it, on reaching the boundary, two things occur.

1. A portion of the energy carried by the pulse is reflected and returns.

2. A portion of the energy carried by the pulse is transmitted to the boundary, causing it to vibrate. But these vibrations are negligible, may contribute to sound or heat and are not discussed here.

The reflected wave gets inverted. That is, if an upward displaced wave is incident towards a fixed end boundary, it will reflect and return as a downward displaced wave and vice versa, as shown in Figure 14.8.

If there is continuous generation of transverse waves from one end and a transmitted wave gets inverted after being reflected from other end which is fixed, the wave interference took place and we get a stationary wave as depicted in Figure 14.8. In our experiment we shall be forming standing waves by continuously generating waves at one end and by keeping the other end fixed.
Figure 14.8: Inversion take place from the fixed boundary.

Resonance

The frequencies at which we get the stationary waves are the natural frequencies of the oscillating system (in our case vibrating string). If we drive one end of the string and when the frequency of the driving force matches with the natural frequency of the string, standing wave is produced and the string begins to move at large amplitude. This phenomenon is called as resonance.

14.5 Wave Speed

Basic definitions

The wavelength $\lambda$ is the distance between two consecutive crests or troughs in case of transverse wave. The period $T$ of the wave is the time required for any particular point on a wave to undergo one complete cycle of transverse or longitudinal motion. The frequency $f$ is the number of the wave cycles completed in one second. The wave number $k$ is the inverse of the wavelength and is expressed as,

$$k = \frac{2\pi}{\lambda}.$$  \hfill (A-2)

Likewise, the angular frequency $\omega$ is,

$$\omega = \frac{2\pi}{T} = 2\pi f.$$  \hfill (A-3)

The length $L$ of the vibrating string in which standing waves are developed can be expressed as integral multiples of half of the wavelength, i.e.,

$$L = \frac{n\lambda}{2},$$  \hfill (A-4)

where $n = 1, 2, 3, \ldots$ is an integer. So, $\lambda$ can be written as,

$$\lambda = \frac{2L}{n}.$$  \hfill (A-5)
Substituting the value of $\lambda$ into Equation A-2, we get,

$$k = \frac{n\pi}{L}. \tag{A-6}$$

The wave speed $v$ can be written as,

$$v = f\lambda = f\frac{2\pi}{k} = \frac{\omega}{k}. \tag{A-7}$$

**Speed of wave on a string subject to tension**

The speed of the waves on a string depends upon the mass of the string element and the tension $T$ under which the string is stretched. The mass of the string element can be expressed in terms of the linear mass density $\mu$, which is the mass per unit length.

![Diagram of string under tension](image)

*Figure 14.9: A small portion of string under the action of forces.*

Let’s consider a small section of string of length $\delta l$ as shown in Figure 14.9. Here $v$ represents the wave speed and the direction in which the wave is traveling is identified by the arrow. The element is part of an arc that is part of an an approximate circle of radius $R$. The mass $\delta m$ of this element is $\mu \delta l$. The tension $T$ in the string is the tangential pull at each end of the segment. The horizontal components cancel since they are equal and opposite to each other. However, each of the vertical components is equal to $T \sin \theta$, so the total vertical force is $2T \sin \theta$. Since $\theta$ is very small, we can approximate $\sin \theta \approx \theta$. Considering the triangle shown in the Figure 14.9, we find that $\delta \theta = \delta l / R$. So, the net force $F$ acting on the string element can be written as,

$$F = 2T \sin \theta \approx 2\theta = T\frac{\delta l}{R}. \tag{A-8}$$
This is the force which is supplying the centripetal acceleration of the string particles towards $O$. The centripetal force $F_c$ acting on the mass $\delta m = \mu \delta l$ moving in a circle of radius $R$ with linear speed $v$ is,

$$F_c = \frac{\delta m v^2}{R} \quad \text{(A-9)}$$

Equating the two forces, we get,

$$T \frac{\delta l}{R} = \frac{\delta m v^2}{R} \quad \text{(A-10)}$$

$$v = \sqrt{\frac{T}{\mu}}. \quad \text{(A-11)}$$

Thus, the wave speed depends upon the tension and the linear mass density of the string.

**Q 2.** Show that Equation A-11 is dimensionally correct.

**Q 3.** A transverse sinusoidal wave is generated at one end of long horizontal string by a bar that moves the end up and down through a distance of $1.5 \text{ cm}$. The motion is repeated at a rate of $130 \text{ times per second}$. If a string has a linear density of $0.251 \text{ kg/m}$ and is kept under a tension of $100 \text{ N}$, find the amplitude, frequency, speed and wavelength of the wave motion.

### 14.6 Experimental setup

Consider a stretched string that is fixed from one end through a rigid support, is strung over a pulley and a weight $W$ is hung at the other end. The string can be set under vibrations using a mechanical oscillator, which in our case is a speaker (woofer) fed with a signal generator. Let $L$ be the length between the wedge and the oscillator as shown in Figure 14.10. If standing waves are established on the string, then the wave vectors can be written as,

$$k_n = \frac{n\pi}{L}, \quad n = 1, 2, 3, \ldots \quad \text{(A-12)}$$

The relation between the angular frequency with the wave vector, is called the dispersion relation and depends upon the effective length and the tension. It can be derived as follows.

Equating Equations A-11 and A-7, we obtain,

$$\frac{\omega}{k} = \sqrt{\frac{T}{\mu}} \quad \text{(A-13)}$$
Hence, for the bare string, the dispersion relation is given by,

$$\omega(k) = \sqrt{\frac{T}{\mu}} \times k = n \sqrt{\frac{T}{\mu} \times \frac{3}{L}}. \quad (A-14)$$

where is \(\omega\) is in radians. The equation clearly shows that that the frequency modes \(\omega(k)\) are directly proportional to the wave vector. Thus the dispersion relation for the string is linear.

A normal mode of an oscillating system is a pattern of motion in which all parts of the system move sinusoidally with the same frequency and in phase. The frequencies of the normal modes are known as its natural frequencies or resonant frequencies. A normal mode is characterized by a mode number \(n\), and is numbered according to the number of half waves in the vibrational pattern. If the vibrational pattern has one stationary wave, the mode number is 1, for two stationary waves, the mode number is 2 and so on.

Our goal in this experiment is to locate the normal modes of the vibrating string. We shall observe the formation of stationary waves on a vibrating string at resonance. The gist of the experiment is that the frequency \(\omega\) will be varied until a pattern of standing waves is observed. When this condition is achieved, there will be an integer number of half-wavelengths formed on the string, which will be vibrating with a large amplitude. This is precisely what a normal mode is! We have excited a normal mode. The driving frequency is in resonance with a normal mode frequency. We will note down the frequency \(\omega\) and the number \(n\) of half-wavelengths and will verify the relationship given in Equation (A-14).

**Procedure**

**Q 4.** After receiving all the equipment, set up the apparatus according to the illustration in Figure 14.10.
Q.5. Carry out the following experimental procedure to find the dispersion relation for the string.

1. Determine the value of \( \mu \) of the string. (The density of the string is 8000 kg/m\(^3\)). What is the uncertainty in the value?

2. Place a driver at any place at the string such that the effective length is 1.5m.

3. Attach 1 kg weight with the other end of the string.

4. Now sweep the frequency slowly using the signal generator and find the frequencies at which the resonance occurs, i.e., where you observe the maximum amplitude standing waves. Start with 1 Hz and increase the frequency with an increment of 0.1 Hz.

5. Note down all the frequencies and plot a graph between frequency and the number of the mode \( n \). This is the desired dispersion relation.

Q.6. Curve fit the data to Equation A-14. What is your calculated value of \( T/\mu \)?

Q.7. Change the string tension using different weights (1.2 kg and 1.4 kg) and plot the dispersion relation for each case. Preferably, plot all your results with varying weight on the same graph. Describe your observations.

Q.8. Change the string length (1 m and 1.25 m) by changing the driver position and plot the dispersion relation for each case. Preferably, plot all your results with varying lengths on the same graph. You should be able to describe your observations.

Circular modes in the vibrating string

The string is being driven in a vertical direction and therefore is expected to vibrate in the vertical plane only. At or near resonance the string has vertical as well as horizontal oscillations. This can be easily visualized by having a side and top view of vibrating string. At this point each element of the string is moving in a circle about the equilibrium position of the string. These oscillatory patterns are called circular modes \([3]\).

Q.9. Is there any change in the string tension as the string vibrates?
14.6. EXPERIMENTAL SETUP

★ Q 10. Can you identify the reason for occurrence of these circular modes?

★ Q 11. Observe what is the effect of decreasing the oscillator's amplitude on the circular modes?

Resonance modes on a loaded string

Theory

The atoms in a crystalline structure are not at rest, they vibrate about their mean positions under the influence of some energy field gradient. This vibrations of the atoms are called lattice vibrations. The energy present in the lattice vibrations can be looked as a series of superimposed sound or strain waves whose frequency spectrum can be determined by the elastic properties of the crystal. The quantum of energy of elastic wave is called phonon. The lattice vibrations can be visualized as a system of identical atoms which are elastically coupled to each other by strings as shown in Figure 14.11.

These lattice vibrations can be simulated experimentally by uniformly loading a string with beads or small masses \( m \) with an inter bead distance \( a \), the resulting dispersion relation would no longer be linear. We will ‘simulate’ the behavior of the chain of the atoms by using a string loaded with rosary beads.

Figure 14.11: Monoatomic lattice vibrations. A simplified picture of a linear chain of atoms.

Experiment

A uniformly loaded string with beads can be used to simulate vibrations in a monoatomic crystalline lattice. Let the total number of beads be \( N \) which are uniformly loaded with an inter bead distance (lattice constant) \( 'a' \). Then the wave vector can be expressed as,

\[
k_n = \frac{n \pi}{(N + 1)a}, \quad n = 1, 2, 3, \ldots, N.
\]  (A-15)
where \( L = (N + 1)a \).

**Q 12.** Carry out the following experimental procedure to find the dispersion relation for the bare string.

1. Measure the interbead distance.
2. Place a driver at the string such that the effective length is 1.5 m.
3. Attach a 1 kg weight with the other end of the string.
4. Now sweep the frequency slowy using signal generator and find the frequency modes at which resonance occurs, i.e., where you observe the maximum amplitude standing waves.
5. Note down all the frequencies and plot a graph between frequency and number of mode \( n \). This is the desired dispersion relation.

**Q 13.** Explain the non linearities in the dispersion relation and any other differences from the bare string.
Bibliography


Chapter 15

Natural Radioactivity and Statistics

Sohaib Shamim, Hafsa Hassan and Muhammad Sabieh Anwar

The discovery of radioactivity was accidental but yet was one of the most astonishing feats of the last few years of the nineteenth century. It has provided experimental evidence influencing many areas of modern physics, such as nuclear and particle Physics. It can also be heralded as the first evidence of the existence of the nucleus- a long series of physics experiments and theoretical innovations finally culminating in the discovery of the weak nuclear force. You have probably studied the basics of radioactivity at school and must be familiar with alpha, beta and gamma radiation, and some of their properties. You must have also heard of the Geiger Counter and the Geiger-Muller (GM) tube. In this experiment you will get a chance to work with radioactive sources and radiation detection apparatus and perform quantitative measurements of their properties.

KEYWORDS

Radioactivity · Background Radiation · Poisson Distribution · Normal/Gaussian Distribution · Data Acquisition.

APPARATUS
15.1 APPROXIMATE PERFORMANCE TIME 1 WEEK

15.2 Conceptual Objectives

In this experiment, we will,

1. learn how to handle radioactive material safely,

2. learn how to show the random statistical nature of radioactivity,

3. practice using Data Acquisition, LabVIEW and mathematical analysis using MATLAB,

4. learn the practical use of the GM Tube and GM counter, and

5. investigate properties of alpha, beta and gamma radiation.

15.3 Experimental Objectives

We will start this experiment with learning how to use the GM Tube, Geiger Counter and data acquisition software with background radiation. We will then demonstrate the random nature of radioactive decay by using the gamma source and fitting our results with the Poisson Distribution. In the following section, we will investigate the penetration properties of the three sources, including the effect of lead absorbers on gamma radiation, and aluminium foil absorbers on beta radiation, and then use our knowledge of the effect of a magnetic field on the three radiations to identify unknown sources.
15.4 Theoretical Introduction

Safety and Radioactivity

The potential dangers of radioactivity are well known, even if they are relatively poorly understood. In this experiment, although we will be mostly dealing with weak sources, it is important to know the precautionary steps that must be imperatively taken, and to have understanding of the dangers involved. The Physlab is authorized by the Pakistan Nuclear Regulatory Authority (PNRA) for the permissible use of radioactive sources for this experiment.

You will be using lead absorbers in this experiment. Lead is a poisonous substance, not safe to be touched by bare hands, so ensure that you wear safety gloves while handling lead.

No eating or drinking is permitted in a radioactivity experiment. Also, any loss of a source can have serious consequences for the institution. The sources for this experiment have been sealed and marked. These should be handed back safely to the supervisor at the end of the experiment.

To get a quantitative idea of the numbers involved, we should study how radioactivity is measured and what powers are considered dangerous. There are three measures of radioactive risk that should be identified and measured [1].

1. The activity of sources is measured in terms of counts/second. 1 Becquerel (Bq) = 1 count per second and 1 Curie (Ci) = $3.7 \times 10^{10}$ Bq. The sources in this experiment have activity of less than 1 $\mu$Ci, i.e., a counting rate of at the most $3.7 \times 10^4$ count per second.

2. The energy and type. Typically, energies released in nuclear decay are measured in MeV. Some of this energy is absorbed before reaching you, and some passes right through you, so only some of this energy can cause biological damage. This will depends on the type of radiation, $\alpha$, $\beta$ or $\gamma$.

3. The lifetime of that source, or in our case, the exposure time, since that is shorter. The biological damage is directly proportional to the duration of exposure to the source.

Q1. Suppose you absorb all the radioactivity from the most active and high energy source over a period of 6 hrs. The count rate is $3.7 \times 10^4$ /s, maximum energy is 1 MeV and your weight is approximately 100 kg. To illustrate the
range of exposures, calculate an upper limit to your exposure as a result of this experiment, in units of Gy [1 Gy = 1 J/kg].

Q 2. Compare this calculation with the maximum permitted yearly dosage of radiation workers, which is of the order $5 \times 10^{-2}$ Gy.

The Poisson Distribution

Since radioactivity is a random process and every decay is independent of the other, we do not expect to get identical number of counts in equal intervals of time. Rather, we obtain varying counts per second every time we repeat the measurement. Mathematically, if something is randomly distributed over time and it is a rare event, then the process can be modelled by a Poisson Distribution, defined by:

$$P(n) = \frac{\mu^n \exp(-\mu)}{n!},$$

where $P(n)$ is the probability of getting $n$ counts in a particular counting interval and $\mu$ is the mean value. The standard deviation is the square root of the mean $\sqrt{\mu}$.

Q 3. The number of particles emitted each minute by a radioactive source is recorded for a period of 10 hours and a total of 1800 counts are registered. During how many 1-minute intervals should we expect to observe

(a) no particles,

(b) 10 particles [2].

One of the main tasks of this experiment is to statistically verify that radioactivity is a random process and follow a Poisson Distribution. To make accurate measurements on low activity sources that we will be using in this experiment, we will need to take into account background radiation first. There are two major sources of background radiation. First is from the natural radioactivity of rocks and minerals on earth. The level from this source depends on the nature of underlying ground and will be higher, for instance, in areas close to granite rocks. The second source is from cosmic radiations. These radiations interact with atoms in the upper atmosphere and create a flux of muons, pions, electrons, neutrons and X-rays. This flux depends on altitude and is of the order of one thousand charged particles per square meter per second near sea level.
15.5 The Apparatus

Geiger Muller (GM) Tube and Counter

The precision Geiger counter manufactured by Daedalon, takes input from the GM tube (also from Daedalon), detects the radiation particles and feeds the signal to the computer. The GM counter clicks every time a radiation particle is detected. The GM tube works best when supplied with 900V. Below this value, its efficiency decreases and we risk losing our data but higher voltage levels can also damage the GM tube.

The sources

The α-source is the isotope of polonium, $^{210}_{84}$Po, which decays with a half life of 133 days. Alpha particles are charged helium nuclei, and have a rest mass energy of about 4000 MeV. These particles have a small kinetic energy when compared with their rest mass energy. This in turn means that their speed is much smaller compared to the speed of light. The small speeds and the large electric charge cause the particles to be absorbed very rapidly as they pass through matter [1].

The β-source is the isotope strontium $^{90}_{38}$Sr with a half-life of 28.6 years. It's strength is about ten times that of the alpha source. Beta decay involves the emission of two particles, a beta particle (electron or positron) and an undetected neutrino (an antineutrino in case of an electron emission). Because the decay
energy is shared between the β-particle and neutrino, there are a number of possible energies for both the emitted β-particle and the neutrino with the kinetic energy of β particles ranging from zero to 0.546 MeV, and the peak occurring at around 0.25 MeV. The β particles have energies close to the speed of light. This combined with a small charge causes them to weakly interact with matter [1].

The γ-source is the isotope cobalt $^{60}$Co, which decays with a half life of 5.26 years. It decays by the emission of two γ rays, with energies close to 1.2 MeV. Before every γ decay, there is a β decay with decay energy of 0.318 MeV, so a single, relatively low energy electron is also emitted in each decay. γ rays of 1.2 MeV and below interact with matter primarily through the Compton Effect and the photoelectric effect, and have weaker interactions than α and β particles [1].

15.6 Experimental Method

Background Radiation

Q 4. Select the Volts mode of the Geiger Counter by using the MODE button, and set the voltage to about 900 V. You should hear distinct clicks from the counter. Open the Labview file Radiation and Statistics Expt located on the desktop. First you are required to record background counts.

Q 5. Before you can start the program, you need to tell the program of the settings you require for the experiment. Choose 300 second intervals for the histogram bin and the sample length. This means that the program will record the number of background clicks for 300 seconds. You are also required to save the data to a file. You should save this file in your Z drive.

Q 6. Click on RUN. Once the acquisition is complete, you need to work out the mean value of background radiation. On the front panel, you can see the total number of counts recorded. Calculate the mean counts/second. Make a note of this in your notebook.

Demonstrating Poisson Distribution with Gamma Radiation

You can now collect the radioactive sources from the instructor. For this part of the experiment, you are required to use the γ-source.
Figure 15.2: Schematic diagram of the experimental set-up used to find the absorption coefficient of different materials.

Q 7. Place the $\gamma$-source such that the mean counts appearing on the Labview program is about 10 counts/second. Record the distance from the GM counter to the source in your notebooks.

Q 8. Now choose 10 second intervals for the histogram bin, 600 seconds (10 minutes) for the sample length. You are also required to enter a path and filename for data to be saved. Make sure you also write down all this information in your lab notebook.

Q 9. Now click RUN. As the program proceeds, you can see a Poisson Distribution graph developing on your computer screen. The next task will be to find the mean and standard deviation of this distribution.

Q 10. Once the experiment is complete, open MATLAB. You can import all the data in Radiation.txt by typing `nucedat` in MATLAB prompt. You can calculate the total number of counts, mean, and standard deviation by using the following commands:

<table>
<thead>
<tr>
<th>Desired variable</th>
<th>Matlab command</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total number of counts</td>
<td><code>sum(counts)</code></td>
</tr>
<tr>
<td>Mean value of counts/second</td>
<td><code>mean(counts)</code></td>
</tr>
<tr>
<td>Standard Deviation of counts/second</td>
<td><code>std(counts)</code></td>
</tr>
</tbody>
</table>

Note the answers to all the variables in your notebooks.

Q 11. Now we will try to generate a theoretical Poisson Distribution with the calculated variables. Use Eq(A-1) and the variables just computed above to generate a theoretical Poisson Distribution in MATLAB.

Q 12. On the same graph plot the histogram for the experimental data.

Q 13. Comment on the two sets of data.
Properties of nuclear radiation

You are provided with a thick paper, aluminium and lead absorbers. You are required to choose suitable absorbers to investigate penetration of $\alpha$, $\beta$ and $\gamma$-sources. Click the large mode button on the Front Panel to switch to the absorption mode.

**Q 14.** Set up the apparatus as shown in Figure 15.2. Place the detector on a marked paper for better measurements of distance $d$.

**Q 15.** Because there are other sources present in the neighborhood, there is likely to be some 'noise' signals from these sources. How would you eliminate this effect?

**Q 16.** Comment on your results for the $\alpha$-source.

**Q 17.** For the $\beta$ and $\gamma$-sources, choose a suitable value of $d$ that gives you around 10 counts per second and plot your data fitting it to the following equation,

$$I = I_0 e^{-\lambda n} \quad \text{(A-2)}$$

where $I_0$ and $I$, represent the initial and final intensity of the sources, $\lambda$ and $n$ represent the absorption coefficient and number of absorbers of the material.

**Q 18.** Calculate the value of $\lambda$ and the corresponding error in it. What are the units?

Effects of Magnetic Fields on Beta Radiations

In this part you will pass the $\beta$ particles through a magnetic field. The field should cause the particles to curve around in a circular path. We will use this information to determine the velocity of $\beta$-particles.

**Q 19.** Setup the apparatus as shown in Figure 15.3. The Helmholtz coil produces a magnetic field into the plane of the paper. The $\beta$-source should be placed very close to the Helmholtz coil. Suggest why this so.

**Q 20.** Rotate the GM tube with the help of available protractor to carry out a 180° scan for the detected particles. Vary the angle by approximately 30° after every reading.

**Q 21.** Plot a graph of angle versus count rate.
Figure 15.3: Schematic diagram of the experimental set-up used to find the effect of magnetic field on Beta Radiations.

Q 22. The magnetic field for the Helmholtz coil is approximately 80 mT. Use this information to determine the minimum, maximum and most probable velocity of β-particles.

15.7 Experience Questions

1. We can also use the gamma source to demonstrate how the counts recorded change with distance (when we were ignoring any absorption). Why would the alpha or Beta source be unsuitable for this purpose?

2. We investigated the effect of absorbers with different sources. Try to explain why the different placement of the same absorber gives different results. There is a simple mathematical explanation for it. Discuss this with the supervisor.

3. Suppose you are given a black box with an unknown number of lead sheets (each of same, certain thickness) inside it. Can you suggest how you could determine the exact number of lead sheets present, using what you have learnt in this experiment so far?

4. Can you think of an experiment to determine the nature of background radiation, i.e., whether its mainly alpha, beta or gamma? (Hint: You can make good use of the absorbers.)
15.8 Idea Experiments

1. We used only typical absorbers like Aluminium, air and lead. Try investigating the effects of the different radiations by absorbers of paper, plastic, wood, other metals etc.

2. Poisson Distribution was used to show the random nature of radioactivity. Another important characteristic of Radioactivity is that it is spontaneous. Can you think of an experiment which can show the spontaneity of radiation?

3. We can argue that for no absorption, the counts recorded fall with distance as $1/r^2$. This is because the radiation from a point source spreads spherically in all direction - a concept that you will learn in your Electricity and Magnetism course next semester. Do you think this would still be the case if our sources could not be approximated as point sources? Suppose you carried out Isq curve fitting with MATLAB and discover that you do not get a $1/r^2$ relation with distance. What can you conclude about your source?

4. Do you think the angle of the absorber sheet with the line perpendicular to the source and detector makes a difference in the counts recorded? Try it out.

5. For demonstrating the Poisson Distribution, we simply plotted the theoretical Poisson plot on top of our experimental data histogram. In fact, there are mathematical ways of checking whether a given distribution satisfactorily fits Poisson (or in general, any given) distribution. Search the $\chi^2$ (Chi squared) goodness of fit tests, and explain how you can use it to check whether your experimental data actually follows a Poisson Distribution. You can also discuss this with the supervisor.

6. It can also be shown that the errors in the means of bins recorded (take the mean of all the means recorded, the error is each mean's deviation from the overall mean) follow a Normal Distribution, given that the number of mean bin counts that you are using is large (over 50). Can you suggest how you could carry out this experiment, using the LabVIEW software given?
Bibliography
