Assignment 4: Solution

1. (a) Consider a one-dimensional diatomic chain of atoms, in which the unit cell is composed of two atoms of same mass $m$, with different force constants between the alternate atoms $\alpha$ and $\beta$ see Fig(1). The spacing between adjacent atoms is $a$, so the unit cell size is $2a$.

The force equation for the $2n$’th atom is,

$$ m \frac{d^2u_{2n}}{dt^2} = -\alpha(u_{2n} - u_{2n+1}) - \beta(u_{2n} - u_{2n-1}), $$  

and for the $(2n + 1)$’th atom,

$$ m \frac{d^2u_{2n+1}}{dt^2} = -\alpha(u_{2n+1} - u_{2n}) - \beta(u_{2n+1} - u_{2n+2}). $$

(b) Assuming the traveling wave solutions to the force equations,

$$ u_{2n} = A_1 e^{i(2na-\omega t)}, $$

$$ u_{2n+1} = A_2 e^{i(2(n+1)a-\omega t)}, $$

$$ u_{2n+2} = A_1 e^{i(2(n+2)a-\omega t)}. $$

Inserting into Eq (1),

$$ m \frac{d^2u_{2n}}{dt^2} = - (\alpha + \beta)u_{2n} + \alpha u_{2n+1} + \beta u_{2n-1} $$

$$ m(-\omega^2)A_1 e^{i(q2na-\omega t)} = - (\alpha + \beta)A_1 e^{i(q2na-\omega t)} + \alpha A_2 e^{i(q(2n+1)a-\omega t)} + \beta A_2 e^{i(q(2n-1)a-\omega t)}, $$

by eliminating $e^{i(q2na-\omega t)}$ from both sides,

$$ -m\omega^2A_1 = - (\alpha + \beta)A_1 + \alpha A_2 e^{iqa} + \beta A_2 e^{-iqa} $$

$$ m\omega^2A_1 = (\alpha + \beta)A_1 - \alpha A_2 e^{iqa} - \beta A_2 e^{-iqa}, $$

$$ m\omega^2A_1 - (\alpha + \beta)A_1 = -\alpha A_2 e^{iqa} - \beta A_2 e^{-iqa} $$

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on rearranging,
\[(m\omega^2 - (\alpha + \beta))A_1 + (\alpha e^{iqa} + \beta e^{-iqa})A_2 = 0.\] (3)

Similarly, Eq (2) becomes,
\[m(-\omega^2)A_2 e^{i(q(2n+1)a - \omega t)} = -(\alpha + \beta)A_2 e^{i(q(2n+1)a - \omega t)} + \alpha A_1 e^{i(q2na - \omega t)} + \beta A_1 e^{i(q(2n+2)a - \omega t)},\]
by eliminating \(e^{i(q2na - \omega t)}\) from both sides,
\[m(-\omega^2)A_2 e^{iqa} = -(\alpha + \beta)A_2 e^{iqa} + \alpha A_1 + \beta A_1 e^{i2qa}\]
\[m\omega^2 A_2 = (\alpha + \beta)A_2 - \alpha A_1 e^{-iqa} - \beta A_1 e^{iqa}\]
\[m\omega^2 A_2 - (\alpha + \beta)A_2 = -\alpha A_1 e^{-iqa} - \beta A_1 e^{iqa}\]
rearranging above expression yields,
\[(\alpha e^{-iqa} + \beta e^{iqa})A_1 + (m\omega^2 - (\alpha + \beta))A_2 = 0.\] (4)

Equations (3) and (4) can be written in the matrix form in two unknowns \(A_1\) and \(A_2\) as,
\[
\begin{bmatrix}
  m\omega^2 - (\alpha + \beta) & \alpha e^{iqa} + \beta e^{-iqa} \\
  \alpha e^{-iqa} + \beta e^{iqa} & m\omega^2 - (\alpha + \beta)
\end{bmatrix}
\begin{bmatrix}
  A_1 \\
  A_2
\end{bmatrix} = 0.
\]
since the equations are homogeneous, a nontrivial solution exists only if the determinant of the matrix vanishes. This leads to the secular equation,
\[
\left|\begin{array}{cc}
  m\omega^2 - (\alpha + \beta) & \alpha e^{iqa} + \beta e^{-iqa} \\
  \alpha e^{-iqa} + \beta e^{iqa} & m\omega^2 - (\alpha + \beta)
\end{array}\right| = 0,
\]
which leads to
\[
[(m\omega^2 - (\alpha + \beta))^2 - (\alpha e^{iqa} + \beta e^{-iqa})(\alpha e^{-iqa} + \beta e^{iqa})] = 0
\]
\[m^2\omega^4 - 2m(\alpha + \beta)\omega^2 + (\alpha + \beta)^2 - (\alpha^2 + \beta^2 + 2\alpha\beta e^{2qa} + \alpha\beta e^{-2qa}) = 0
\]
\[m^2\omega^4 - 2m(\alpha + \beta)\omega^2 + (\alpha + \beta)^2 - (\alpha^2 + \beta^2) - 2\alpha\beta \left(\frac{e^{2qa} + e^{-2qa}}{2}\right) = 0
\]
\[m^2\omega^4 - 2m(\alpha + \beta)\omega^2 + (\alpha + \beta)^2 - (\alpha^2 + \beta^2 - 2\alpha\beta \cos(2qa)) = 0
\]
\[m^2\omega^4 - 2m(\alpha + \beta)\omega^2 + (\alpha + \beta)^2 - (\alpha^2 + \beta^2 + 2\alpha\beta) + 2\alpha\beta(1 - \cos(2qa)) = 0
\]
\[m^2\omega^4 - 2m(\alpha + \beta)\omega^2 + (\alpha + \beta)^2 - (\alpha + \beta)^2 + 2\alpha\beta(2\sin^2(2qa/2)) = 0
\]
\[m^2\omega^4 - 2m(\alpha + \beta)\omega^2 + 4\alpha\beta \sin^2(qa) = 0
\]
\[\omega^4 - \frac{2}{m}(\alpha + \beta)\omega^2 + \frac{4\alpha\beta}{m^2} \sin^2(qa) = 0.
\]
This is a quadratic equation in $\omega^2$. It has two roots

$$\omega^2 = \frac{2}{m} (\alpha + \beta) \pm \frac{1}{m} \sqrt{(\alpha + \beta)^2 - 4\alpha\beta \sin^2(qa)}$$

$$\omega^2 = \frac{(\alpha + \beta)}{m} \pm 1 \frac{1}{m} (\alpha + \beta)^2 - 4\alpha\beta \sin^2(qa).$$

(5)

(c) Let $\alpha = (r - 1)\beta$. Substituting in Eq(5), we get,

$$\omega^2 = \frac{(r - 1)\beta + \beta}{m} \pm 1 \frac{1}{m} \sqrt{(r - 1)\beta + \beta}^2 - 4(r - 1)\beta \sin^2(qa)$$

$$\omega^2 = \frac{r\beta + \beta}{m} \pm \frac{r\beta + \beta}{m} \sqrt{1 - \frac{4(r - 1)}{r^2} \sin^2(qa)}$$

$$\omega^2 = \frac{r\beta}{m} [1 \pm \sqrt{1 - \frac{4(r - 1)}{r^2} \sin^2(qa)}].$$

(6)

For $\alpha = 10\beta$, $10\beta = (r - 1)\beta$ or $r = 11$, the dispersion relation is simplified as,

$$\omega^2 = \frac{r\beta}{m} [1 \pm \sqrt{1 - \frac{4(10}{11^2} \sin^2(qa)}]$$

$$\omega^2 = \frac{r\beta}{m} [1 \pm \sqrt{1 - 4\left(\frac{10}{121}\right) \sin^2(qa)}].$$

(7)

$r=11$

Plot[$\{Sqrt[1 + Sqrt[1 - 4 \cdot (r - 1)/r^2 \cdot Sin[\phi]^2]],$

$Sqrt[1 - Sqrt[1 - 4 \cdot (r - 1)/r^2 \cdot Sin[\phi]^2]]\}$,

$\{\phi, -Pi/2, Pi/2\}$, AxesLabel $\rightarrow \{qa, \omega \sqrt{m}/\sqrt{(r\beta)}\}$

Fig(2) : Dispersion relation in first Brillouin zone for $\alpha=10\beta$
Similarly, for $\alpha = 100\beta$, $r = 101$ Eq(6) becomes

$$\omega^2 = \frac{r\beta}{m} \left[ 1 \pm \sqrt{1 - 4 \left( \frac{10}{101^2} \right) \sin^2(qa)} \right]$$

$$\frac{\omega^2 m}{r\beta} = 1 \pm \sqrt{1 - 4 \left( \frac{10}{10201} \right) \sin^2(qa)}.$$  \hspace{1cm} (8)

At $q = 0$, Eq(6) becomes,

$$\omega^2 = \left( \frac{\alpha + \beta}{m} \right) \pm \left( \frac{\alpha + \beta}{m} \right).$$

Corresponding to the two signs in above equation, there are thus two dispersion relations, and consequently two dispersion curves, or branches, associated with the diatomic chain,

Acoustic: \hspace{0.5cm} $\omega_-^2 = \left( \frac{\alpha + \beta}{m} \right) - \left( \frac{\alpha + \beta}{m} \right) = 0 \Rightarrow \omega = 0,$

Optic: \hspace{0.5cm} $\omega_+^2 = \left( \frac{\alpha + \beta}{m} \right) + \left( \frac{\alpha + \beta}{m} \right)

\hspace{1cm} = \frac{2}{m} (\alpha + \beta) \approx \frac{2\alpha}{m} \Rightarrow \alpha \gg \beta.$

Fig(3): Dispersion relation in first Brillouin zone for $\alpha=100\beta$

for $\alpha = 100\beta$, there will be hardly any dispersion in optic branch.

(d) At $q = 0$, Eq(6) becomes,

$$\omega^2 = \left( \frac{\alpha + \beta}{m} \right) \pm \left( \frac{\alpha + \beta}{m} \right).$$

Corresponding to the two signs in above equation, there are thus two dispersion relations, and consequently two dispersion curves, or branches, associated with the diatomic chain,

Acoustic: \hspace{0.5cm} $\omega_-^2 = \left( \frac{\alpha + \beta}{m} \right) - \left( \frac{\alpha + \beta}{m} \right) = 0 \Rightarrow \omega = 0,$

Optic: \hspace{0.5cm} $\omega_+^2 = \left( \frac{\alpha + \beta}{m} \right) + \left( \frac{\alpha + \beta}{m} \right)

\hspace{1cm} = \frac{2}{m} (\alpha + \beta) \approx \frac{2\alpha}{m} \Rightarrow \alpha \gg \beta.$
From Eq(3) and Eq(4), when $\omega^2 = 0$ in the acoustic branch:

$$(0 - (\alpha + \beta))A_1 + (\alpha + \beta)A_2 = 0$$

$$(\alpha + \beta))A_1 = (\alpha + \beta)A_2$$

$\Rightarrow A_1 = A_2.$

Thus for this branch the two atoms in the molecule, have the same amplitude, and are also in phase (see Fig (4)).

Now, in the optic branch at $q = 0$, $\omega^2 = \frac{2m}{m}(\alpha + \beta)$. Therefore, Eq(3) and Eq(4) becomes

$$\left(\frac{2m}{m}(\alpha + \beta) - (\alpha + \beta)\right)A_1 + (\alpha + \beta)A_2 = 0$$

$$(\alpha + \beta)A_1 = -(\alpha + \beta)A_2$$

$\Rightarrow A_1 = -A_2.$

Thus for this branch the two atoms move $\pi$ out of phase with each other. Intermolecular forces are small, thats why there is no dispersion in the optical branch (see Fig (5)).

2. (a) For a 1D monatomic lattice, assuming the interaction of atoms with its nearest neighbors only (see Fig 6), the force equation for the $n$’th atom is
\[ m \frac{d^2 u_n}{dt^2} = \alpha (u_{n+1} - u_n) + \alpha (u_{n-1} - u_n) \]
\[ = \alpha (u_{n+1} + u_{n-1} - 2u_n). \]

The potential energy of the crystal system, modeled as a linear chain of atoms is given by

\[ U_{P,E} = \frac{1}{2} \alpha \sum_n x_n^2, \]
where \( n \) runs over all the atoms. Therefore, the potential energy of the \( n \)'th atom is,

\[ U = \frac{1}{2} \alpha \sum_n (u_{n+1} - u_n)^2, \tag{9} \]
where \((u_{n+1} - u_n) = -x\) is the relative displacement of the \( n \)'th atom. Differentiating Eq(9) w.r.t \( x \) yields,

\[ \frac{dU}{dx} = \frac{1}{2} \alpha \sum_n \frac{d(x^2)}{dx}, \]
\[ = \alpha \sum_n (u_n - u_{n+1}). \]

The force and potential energy of the harmonic oscillator are related by the following expression,

\[ F_n = -\frac{dU}{dx} \]
\[ = -\alpha \sum_n (u_n - u_{n+1}) \]
\[ = -(\alpha (u_n - u_{n+1}) + \alpha (u_n - u_{n-1})) \]
\[ = \alpha (u_{n+1} + u_{n-1} - 2u_n) \]
\[ \Rightarrow m \frac{d^2 u_n}{dt^2} = F_n = -\alpha (2u_n - u_{n+1} - u_{n-1}) \]

Since the atom is interacting with \( p \) atoms on either side, therefore we can write the above expression as,

\[ m \frac{d^2 u_n}{dt^2} = F_n = -\sum_{\pm p} \alpha_p (u_n - u_{n+p}). \]

(b) For the second nearest neighbor atoms, we have,

\[ p = 2 \quad \text{and} \quad \alpha_1 = \alpha_2. \]
The force equation for the $n$’th atom, interacting with four neighboring atoms (see Fig 7), is

$$m \frac{d^2 u_n}{dt^2} = -\alpha \left( 4u_n - u_{n+1} - u_{n-1} - u_{n+2} - u_{n-2} \right).$$  \hspace{1cm} (10)$$

Assume the traveling wave solution to force equation,

$$u_n = A e^{i(qna-\omega t)},$$

$$u_{n+p} = A e^{i(q(n+p)a-\omega t)}.$$  

inserting in Eq (10), we get

$$-\omega^2 mA e^{i(qna-\omega t)} = -\alpha A \left[ 4e^{i(qna-\omega t)} - e^{i(q(n+1)a-\omega t)} - e^{i(q(n-1)a-\omega t)} - e^{i(q(n+2)a-\omega t)} - e^{i(q(n-2)a-\omega t)} \right],$$

eliminate $A e^{i(qna-\omega t)}$ from both sides,

$$-\omega^2 m = -\alpha \left[ 4 - e^{iqa} - e^{-iqa} - e^{i2qa} - e^{-i2qa} \right]$$

$$\omega^2 m = \alpha \left[ 4 - 2 \left( \frac{e^{iqa} + e^{-iqa}}{2} \right) - 2 \left( \frac{e^{i2qa} + e^{-i2qa}}{2} \right) \right]$$

$$\omega^2 m = \alpha [4 - 2 \cos(qa) - 2 \cos(2qa)]$$

$$\Rightarrow \omega^2 = \left( \frac{2\alpha}{m} \right) [(1 - \cos(qa)) + (1 - \cos(2qa))]$$

$$= \left( \frac{4\alpha}{m} \right) \left[ \sin^2 \left( \frac{qa}{2} \right) + \sin^2(qa) \right]$$

$$= \left( \frac{4\alpha}{m} \right) \left[ \sin^2 \left( \frac{qa}{2} \right) + 4 \sin^2 \left( \frac{qa}{2} \right) \cos^2 \left( \frac{qa}{2} \right) \right],$$

where $\sin \theta = 2 \sin(\theta/2) \cos(\theta/2),$

$$\Rightarrow \omega^2 = \left( \frac{4\alpha}{m} \right) \left( \sin^2 \left( \frac{qa}{2} \right) \right) \left[ 1 + 4 \cos^2 \left( \frac{qa}{2} \right) \right],$$

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which is the required relation.

(c) \( \text{Plot}\{\sqrt{\sin(\phi/2)^2}, \sqrt{\sin(\phi/2)^2 \cdot (1 + 4 \cdot \cos(\phi/2)^2)}\}, \{\phi, -\pi, \pi\}, \text{AxesLabel} \rightarrow \{k_a, \omega \sqrt{m/2\sqrt{\alpha}}\} \)

3. (a) For the 1D, monatomic lattice of length \( L \) (see Fig 9), in the long wavelength limit, corresponding to \( q \to 0 \), the linear dispersion relation holds good

\[
\omega = vq
\]

\[
= v \frac{2n\pi}{L},
\]

where \( n = 0, \pm 1, \pm 2, \pm 3 \), etc are allowed \( q \) values (see Fig 9),

\[
\Rightarrow d\omega = v dq = v \frac{2\pi}{L} dn.
\]
The number of modes in an interval $dq$ are

$$\begin{align*}
dN &= 2 \times \frac{dq}{\text{length per mode in } q\text{-space}} \\
&= 2 \times \frac{dq}{2\pi/L} \\
&= \frac{Ldq}{\pi} \\
&= \frac{L d\omega}{\pi v}.
\end{align*}$$

Since the density of modes $g(\omega)$ is the number of modes per unit frequency range $d\omega$,

$$g(\omega) = \frac{dN}{d\omega} = \frac{L}{\pi v} = \frac{L}{\pi(d\omega/dq)}.$$

For monatomic lattice, the normal mode frequency is

$$\omega = \omega_m |\sin(qa/2)|,$$

where $\omega_m = (4\alpha/m)^{1/2}$,

$$\Rightarrow \frac{d\omega}{dq} = \left(\frac{4\alpha}{m}\right)^{1/2} \frac{a}{2} \cos(qa/2).$$

Therefore, density of modes is

$$g(\omega) = \frac{L}{\pi \left(\frac{4\alpha}{m}\right)^{1/2} \frac{a}{2} \cos(qa/2)} = \frac{L}{\pi a \omega_m \cos(qa/2)}.$$

The total number of modes $N$ for 1D monatomic lattice is

$$\int_0^{\omega_m} g(\omega)d\omega = \int_0^{\omega_m} \frac{L}{\pi} \left(\frac{dq}{d\omega}\right) d\omega,$$

where $\omega_m$ corresponds to $q = \pi/a$. 

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The dispersion curve  Fig (10): $\omega$ versus $q$

\[
\int_0^{\omega_m} g(\omega) d\omega = \frac{L}{\pi} \int_0^{\pi/a} dq = \frac{L \pi}{a} \\
\Rightarrow \int_0^{\omega_m} g(\omega) d\omega = \frac{L}{a} = N.
\]

Plot[$1/(1 - x^2), \{x, 0, 1\}$, AxesLabel -> {$\omega, g(\omega)\pi\omega_m a/(2L)$}]

Fig (11): Density of states for a 1D lattice

Discrete lattice
Continuum lattice

4.
THERMAL CONDUCTIVITY OF SOLIDS

The thermal conductivity coefficient $K$ of a solid is most easily defined with respect to the steady state flow of heat down a long rod within which there exists a temperature gradient $\partial T/\partial x$:

$$Q = K \frac{\partial T}{\partial x},$$

where $Q$ is the flux of thermal energy (energy transmitted across unit area per unit time); $K$ is often expressed in units cal/cm-sec-deg or watts/cm-deg. To convert to watts/cm-deg, multiply $K$ in cal/cm-sec-deg by 4.186.

The form of the equation (6.41) defining the conductivity implies that the process of thermal energy transfer is a random process. The energy does not simply enter one end of the specimen and proceed directly in a straight path to the other end, but the energy diffuses through the specimen, suffering frequent collisions. If the energy were propagated directly through the specimen without deflection, then the expression for the thermal flux would not depend on the temperature gradient, but only on the difference in temperature $\Delta T$ between the ends of the specimen, regardless of the length of the specimen. It is the random nature of the conductivity process that brings the temperature gradient into the expression for the thermal flux.

In gas-kinetic theory one finds in a certain approximation the following expression for the thermal conductivity:

$$K = \frac{1}{3}Cu\Lambda,$$

where $C$ is the heat capacity per unit volume, $u$ is the average particle velocity, and $\Lambda$ is the mean free path of a particle between collisions. This result is derived in the following section. The result was applied first by Debye to describe thermal conductivity in dielectric solids, $C$ being the heat capacity of the lattice waves or phonons, $u$ the velocity of sound, and $\Lambda$ the mean free path of the phonons. Several representative values of the mean free path are given in Table 6.5.
Table 6.5. Phonon Mean Free Path Values

[Calculated from (6.42), taking $u = 5 \times 10^5$ cm/sec as the average sound velocity]

<table>
<thead>
<tr>
<th>Crystal</th>
<th>$T$(°C)</th>
<th>$C$(cal/cm$^3$-deg)</th>
<th>$K$(cal/cm-deg-sec)</th>
<th>$\Lambda$(cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quartz†</td>
<td>0</td>
<td>0.48</td>
<td>0.03</td>
<td>40 $\times$ 10$^{-8}$</td>
</tr>
<tr>
<td></td>
<td>-190</td>
<td>0.13</td>
<td>0.12</td>
<td>540 $\times$ 10$^{-8}$</td>
</tr>
<tr>
<td>NaCl</td>
<td>0</td>
<td>0.45</td>
<td>0.17</td>
<td>23 $\times$ 10$^{-8}$</td>
</tr>
<tr>
<td></td>
<td>-190</td>
<td>0.24</td>
<td>0.064</td>
<td>100 $\times$ 10$^{-8}$</td>
</tr>
</tbody>
</table>

† Parallel to optic axis.

Calculation of Thermal Conductivity

We give now the kinetic theory calculation of the thermal conductivity of a classical electron gas. We consider the transfer of energy by electrons crossing the $xy$ plane; the temperature gradient is along the $z$ axis. An electron traveling a distance equal to the mean free path $\Lambda$ and striking the plane at a polar angle $\theta$ has a mean energy

$$E(0) + (\Lambda \cos \theta) \frac{\partial E}{\partial z},$$

where $E(0)$ is the mean energy at the plane $z = 0$. By a well-known kinetic theory result the number of molecules which cross unit area of the plane in a direction making an angle between $\theta + d\theta$ with the $z$ axis per second is

$$\frac{1}{2}Nu \cos \theta \sin \theta d\theta,$$

where $N$ is the concentration and $u$ is the average velocity. The net energy flux is

$$(6.43) \quad \frac{1}{2} Nu \frac{\partial E}{\partial z} \int_0^\pi \Lambda \cos^2 \theta \sin \theta d\theta = \frac{1}{3} Nu \Lambda \frac{\partial E}{\partial z},$$

which, by the definition of the thermal conductivity $K$, must be equal to $K(\partial T/\partial z)$. Now

$$(6.44) \quad N(\partial E/\partial z) = N(\partial E/\partial T)(\partial T/\partial z) = C(\partial T/\partial z),$$

where we have introduced the relaxation time $\tau = \Lambda/u_F$. 
It is instructive to bring out the random nature of the conductivity process. The usual heat conduction equation for the flux of thermal energy in a linear specimen is

\[(6.48) \quad Q = K(T_1 - T_2)/L.\]

Here \(Q\) = heat transfer across unit area per unit time; \(T_1 - T_2\) = temperature drop between ends; \(L\) = length of specimen. On substituting for \(K\) the expression given by \((6.45)\), we have

\[(6.49) \quad Q = \frac{1}{3}C(T_1 - T_2)(\Delta u/L).\]

In this form the equation may be interpreted very simply: \(C(T_1 - T_2)\) is the excess energy density at one end of the specimen with respect to the other end; this excess is propagated down the specimen with an effective transport velocity which is just the carrier velocity reduced by the ratio of the mean free path to the length of the specimen.\(^8\)

5. The volume of the reciprocal lattice unit cell is related to the real lattice unit cell by the following expression,

\[ V_k = \frac{(2\pi)^3}{V}, \quad (11) \]

where \( V_k \) is the volume of the first Brillouin zone and \( V \) is the volume of the real lattice unit cell.

The volume of the first Brillouin zone is given as,

\[ V_k = \frac{4}{3} \pi q_d^3, \quad (12) \]

Comparing Equations (11) and (12),

\[ \frac{4}{3} \pi q_d^3 = \frac{(2\pi)^3}{V}, \]

on rearranging,

\[ q_d = 2\pi \left( \frac{3}{4\pi V} \right)^{1/3} \quad (13) \]

The dispersion relation in the elastic limit is,

\[ \omega_d = v_s q_d. \]

Substituting Equation (13) in the above equation gives,

\[ \omega_d = 2\pi v_s \left( \frac{3}{4\pi V} \right)^{1/3} \quad (14) \]

The primitive translational vectors of the hexagonal Bravais lattice are,

\[ \vec{a} = a\hat{x}, \]
\[ \vec{b} = a\left(-\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\right), \]
\[ \vec{c} = c\hat{z}. \]

The volume of the hexagonal Bravais lattice is,

\[ V = \vec{a} \cdot (\vec{b} \times \vec{c}). \quad (15) \]
Now,

\[ \vec{b} \times \vec{c} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{a}{2} & \frac{a\sqrt{3}}{2} & 0 \\ 0 & 0 & c \end{vmatrix} , \]

\[ = \left( \frac{ac\sqrt{3}}{2} \right) \hat{x} + \left( \frac{ac}{2} \right) \hat{y} . \]

Substituting the above expression in Equation (15) yields,

\[ V = \vec{a} \cdot (\vec{b} \times \vec{c}) = \frac{a^2 c\sqrt{3}}{2} . \]

The expression (14) becomes,

\[ \omega_d = 2\pi v_s \left( \frac{3}{4\pi} \right)^{1/3} \left( \frac{2}{a^2 c\sqrt{3}} \right)^{1/3} \]

\[ = 2\pi v_s \left( \frac{\sqrt{3}}{2\pi a^2 c} \right)^{1/3} . \]