Moder Phys Recitation

First order differential equation

General form

\[ \frac{dy}{dx} = f(x, y) \] or \[ \frac{dx}{dt} = f(x(t)) \]

Problem: Describe a system undergoing radioactivity

Let \( x(0) = N_0 \) = initial number of radioactive nuclei in the sample.

To find \( x(t) \) for any later time given that the material has an "activity" of magnitude \( \lambda \).

Rate of change in radioactive nuclei \( \lambda \) = no. of radioactive nuclei present in the sample.

\[ \frac{dx}{dt} \propto \lambda \]

As the quantity \( x(t) \) decreases with time.

\[ \frac{dx}{dt} = -\lambda x \]

\[ \int \frac{dx}{x} = -\lambda \int dt \]

\[ \ln x + c_1 = -\lambda t + c_2 \]

\[ \ln x = -\lambda t + c_3 \] \[ \Rightarrow A \] where \( c_3 = c_2 - c_1 \)
taking exponential on both sides

\[ x(t) = C e^{-\lambda t} \]

For \( t = 0 \) \( x(0) = C e^0 = C \)

Second order differential equations

General form (homogeneous)

\[ a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0 \]

System: Mass attached to a spring

(Simple Harmonic Motion)

Neglect damping

(friction, air drag etc)

\[ F = -kx \]

\[ m \frac{d^2 x}{dt^2} = -kx \]

\[ m x'' + kx = 0 \]
Suppose \( x(t) = e^{rt} \), \( x(t) = ye^{rt} \), \( x''(t) = r^2 e^{rt} \).

Even becomes
\[
mr^2 e^{rt} + ke^{rt} = 0
\]
\[
(mr^2 + k)e^{rt} = 0 \Rightarrow mr^2 + k = 0
\]

Characteristic equation

Two roots \( r = \pm i\sqrt{k/m} \) and \( r_2 = -i\sqrt{k/m} \)

Define \( \omega \equiv \sqrt{k/m} \Rightarrow r_1 = +i\omega \) and \( r_2 = -i\omega \)

General solution is then
\[
x(t) = (c_1 e^{i\omega t} + c_2 e^{-i\omega t})
\]

Euler's identity.
\[
e^{i\omega t} = \cos \omega t + i \sin \omega t.
\]
\[
e^{-i\omega t} = \cos \omega t - i \sin \omega t.
\]
\[
e^{i\omega t} + e^{-i\omega t} = 2 \cos \omega t \Rightarrow \cos \omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}
\]
\[
e^{i\omega t} - e^{-i\omega t} = 2i \sin \omega t \Rightarrow \sin \omega t = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}
\]

Can take the following forms
\[
x_{g_1}(t) = \frac{1}{2} e^{i\omega t} + \frac{1}{2} e^{-i\omega t} = \cos \omega t.
\]
\[
x_{g_2}(t) = \frac{1}{2i} e^{i\omega t} - \frac{1}{2i} e^{-i\omega t} = \sin \omega t.
\]

The most general solution is then
\[
x(t) = A \cos \omega t + B \sin \omega t
\]
\[ x(t) = D \left[ \cos \theta \cos \omega t - \sin \theta \sin \omega t \right] \]

\[ x(t) = D \cos (\omega t + \theta) \]

Depending on \( \theta \), the solution can have sine or cosine or mixed form.

**Damped Harmonic Oscillator**

Let the damping is denoted by \( \dot{d} \) then the damping term in the equation is a multiple of velocity term \( \frac{dx}{dt} \) or \( x' \).

\[ m x'' + \dot{d} x' + k x = 0 \] \( \rightarrow \) (A)

Assuming \( x = e^{rt} \), \( x' = re^{rt} \), \( x'' = r^2 e^{rt} \)

\[ mr^2 + \dot{d} r + k = 0 \] \( \rightarrow \) characteristic equation.

Binomial roots:

\[ r = \frac{-\dot{d} \pm \sqrt{\dot{d}^2 - 4mk}}{2m} \]

If \( \dot{d}^2 > 4mk \) \( \rightarrow \) over-damped case (real and distinct roots)

\( \dot{d}^2 = 4mk \) \( \rightarrow \) critically-damped case (identical roots)

\( \dot{d}^2 < 4mk \) \( \rightarrow \) under-damped case (imaginary roots)
Consider lower damping
\[ d^2 < 4mk. \]

\[
\Rightarrow \sqrt{d^2 - 4mk} = i \sqrt{4mk - d^2}
\]

\[ r = -\frac{d}{2m} + \frac{i \sqrt{4mk - d^2}}{2m} \]

\[ \alpha = \frac{d}{2m} \quad \beta = \frac{\sqrt{4mk - d^2}}{2m} \]

\[ r_1 = -\alpha + i\beta \quad \text{and} \quad r_2 = -\alpha - i\beta. \]

The solution is then
\[ x(t) = Ae^{-\alpha t} e^{i\beta t} + Be^{-\alpha t} e^{-i\beta t}. \]

\[ x(t) = e^{-\alpha t} \left[ c_1 e^{i\beta t} + c_2 e^{-i\beta t} \right] \]

which can be written as
\[ x(t) = e^{-\alpha t} \left[ A \cos \beta t + B \sin \beta t \right] \]

where
\[ A = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \]
\[ B = \frac{c_2}{\sqrt{c_1^2 + c_2^2}} \]
\[ X(t) = e^{-\alpha t} + Be^{i\beta t} + Be^{-i\beta t} \]