1. An atom has two energy levels with a transition wavelength of 580 nm. At room temperature $4 \times 10^{20}$ atoms are in the lower state.

(a) How many atoms occupy the upper state, under conditions of thermal equilibrium?

(b) Suppose instead that $7 \times 10^{20}$ atoms are pumped into the upper state, with $4 \times 10^{20}$ in the lower state. This is clearly a non-equilibrium state. How much energy in Joules could be released in a single pulse of light as equilibrium is restored?

The Boltzmann factor is $\exp(-E/k_B T)$ where $k_B = 1.38 \times 10^{-23}$ J/K.

**Answer 1:**

If $N_b$ is the higher energy state and $N_a$ is the lower energy state, then the energy difference between these two states can be calculated using transition wavelength of 5800 Å, i.e.,

$$E_b - E_a = hf = \frac{hc}{\lambda}$$

$$= \frac{(6.63 \times 10^{-34} \text{ Jsec})(3 \times 10^8 \text{ m/sec})}{5800 \times 10^{-10} \text{ m}}$$

$$E_b - E_a = 3.43 \times 10^{-19} \text{ J}$$

$$k_B T = 1.38 \times 10^{-28} \text{ J/K} \times 300 \text{ K}$$

$$= 4.1 \times 10^{-21} \text{ J}.$$

According to the Boltzmann distribution, the population in any state $i$ is,

$$N_i = e^{-E_i/k_B T}.$$

Therefore, the ratio of $N_a$ and $N_b$ in equilibrium is,

$$\frac{N_{b}^{\text{eq}}}{N_{a}^{\text{eq}}} = e^{-(E_b - E_a)/k_B T} = e^{-(3.43 \times 10^{-19} \text{ J}/4.1 \times 10^{-21} \text{ J})}$$

$$= 1.1 \times 10^{-36}$$

$$N_{b}^{\text{eq}} = N_{a}^{\text{eq}} \times 1.1 \times 10^{-36}$$

$$= (4 \times 10^{20})(1.1 \times 10^{-36})$$

$$= 4.4 \times 10^{-16},$$
which is very small. This shows that the energy gap between the given states $N_a$ and $N_b$, at room temperature, is so large that almost all the electrons are present in lower state $N_a$. There is hardly any electrons in the upper state.

(b) When atoms are pumped in the upper state, this is a non-equilibrium situation. Energy is released up till equilibrium is restored. Therefore,

$$
N_{\text{non-eq}}^a = 4 \times 10^{20} \\
N_{\text{non-eq}}^b = 7 \times 10^{20} \\
N_{\text{non-eq}}^c = N_{\text{non-eq}}^a + N_{\text{non-eq}}^b \\
= 11 \times 10^{20}.
$$

We are interested in number of atoms which restores the equilibrium. Since the ratio, $N_a/N_b = 1.1 \times 10^{-36}$, remains the same (the gap width does not change), therefore,

$$
N_{\text{non-eq}}^a + N_{\text{non-eq}}^b = 11 \times 10^{20} \\
N_{\text{non-eq}}^b(1 + 9.09 \times 10^{-35}) = 7 \times 10^{20} \\
N_{\text{non-eq}}^b = 1.21 \times 10^{-15}.
$$

The number of atoms which contribute to restore equilibrium is,

$$
\Delta N = N_{\text{non-eq}}^a - N_{\text{non-eq}}^b \\
\approx 7 \times 10^{20}.
$$

The energy released in a single pulse is thus,

$$
E = \Delta N \times \frac{hc}{\lambda} \\
= 7 \times 10^{20} \times 3.43 \times 10^{-19} \text{ J} \\
= 240 \text{ J}.
$$

2. Suppose a proton and an electron were held together in a hydrogen atom by gravitational forces only.

(a) Find the formula for the energy levels of such an atom, and the radius of the ground state Bohr orbit.
(b) Find the numerical value of the Bohr radius for this “gravitational” atom.

Reduced Planck’s constant $\hbar = 1.06 \times 10^{-34}$ Js, Mass of proton $M = 1.67 \times 10^{-27}$ kg, Mass of electron $m = 9.11 \times 10^{-31}$ kg, Gravitational constant $G = 6.67 \times 10^{-11}$ Nm$^2$/kg$^2$.

**Answer 2:**

Let $r$ be the radius of electron’s orbit, then gravitational force between proton and electron is,

$$F_g = G \frac{mM}{r^2}.$$  \hspace{1cm} (1)

Centripetal force needed to keep the electron in a circular orbit is provided by this gravitational force. If $v$ is the orbital speed of electron, centripetal force will be,

$$F_c = \frac{mv^2}{r}.$$  \hspace{1cm} (2)

Comparing equations (1) and (2), we obtain,

$$\frac{mv^2}{r} = G \frac{mM}{r^2}$$

$$v^2 = G \frac{M}{r}.$$  \hspace{1cm} (3)

According to Bohr atomic model “the allowed orbits are those for which the electron’s orbital angular momentum about the nucleus is an integral multiple of $\hbar$”, i.e.,

$$mvr = n\hbar \quad n = 1, 2, 3, \ldots$$

$$\Rightarrow v = \frac{n\hbar}{mr}$$

$$v^2 = \frac{n^2\hbar^2}{m^2r^2}.$$  \hspace{1cm} (4)

Comparing equations (3) and (4),

$$\frac{GM}{r} = \frac{n^2\hbar^2}{m^2r^2}$$

$$\Rightarrow r = n^2 \frac{\hbar^2}{GMm^2},$$

which is the required formula for the radius of the Bohr orbit. For the ground state, set $n = 1$. Total energy of the atom, which consists of both the kinetic and potential energy terms is,

$$E = \frac{1}{2}mv^2 - G \frac{Mm}{r}.$$
Substitute value of $v^2$ from equation (3),

\[ E = \frac{1}{2}m\left(\frac{GM}{r}\right) - G\frac{Mm}{r} \]

\[ = -\frac{GMm}{2} \cdot \frac{1}{r} \]

\[ = -\frac{GMm}{2} \cdot \frac{1}{n^2 \frac{h^2}{GMm^2}} \]

\[ = -\frac{GMm}{2} \cdot \frac{GMm^2}{n^2 h^2} \]

\[ E = -\frac{G^2 M^2 m^3}{2n^2 h^2}, \]

which is the required formula for energy levels.

(b) Since,

\[ r_n = n^2 \frac{h^2}{GMm^2} \]

For $n = 1$,

\[ r_1 = \frac{h^2}{GMm^2}. \]

Substitution of the given values yields,

\[ r_1 = \frac{(1.06 \times 10^{-34} \text{ Js})^2}{6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2 \times 1.67 \times 10^{-27} \text{ kg} \times (9.11 \times 10^{-31} \text{ kg})^2} \]

\[ = 1.2 \times 10^{20} \text{ m.} \]

Note $r_1$ is a huge value, because gravitational attraction is much much weaker than electrostatic interaction. Even though this is not required, we calculate the energy in the lowest orbit. Given

\[ E_n = -\frac{G^2 M^2 m^3}{2n^2 h^2}, \]

and setting $n = 1$,

\[ E_1 = -\frac{G^2 M^2 m^3}{2h^2} \]

\[ = -(6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2)^2 \times \frac{(1.67 \times 10^{-27} \text{ kg})^2 \times (9.11 \times 10^{-31} \text{ kg})^3}{2(1.06 \times 10^{-34} \text{ Js})^2} \]

\[ = -4.17 \times 10^{-97} \text{ J}. \]