Solution: Midterm Exam

1. The period T of a rigid pendulum is determined by measuring the time t taken for an integral number of swings N. The error Δt in t comes from starting and stopping the timer and may be assumed to be independent of t. So the larger the value of N and hence of t, the more precise is the value of the period.
   Let the value of Δt be 0.2 s. 20 swings are counted and are found to take a time $t = 40.8$ s. The pendulum is set swinging again; this time the swings are not counted, but an integral number N_t are found to take 162.9 s. Deduce the values of N_t and the final value of T with its uncertainty. (Assume that the amplitude of the swings is sufficiently small for the variation in the period to be negligible throughout the measurements). (3 points)

   (a) $N_t = 79$, $T = 2.062 \pm 0.002$ s.

   (b) $N_t = 80$, $T = 2.036 \pm 0.002$ s.

   (c) $N_t = 81$, $T = 2.011 \pm 0.002$ s.

   (d) None of the above.

   Solution:

   The correct answer is (**b**).

   For 20 swings, time is measured as,

$$t = (40.8 \pm 0.2),$$

   the time period of the pendulum can be found out,

$$T = \frac{t}{20} = 2.04\,\text{s}.$$

   and uncertainty can be calculated by using the Taylor series approximation,

$$\begin{aligned}\Delta T &= \sqrt{\left(\frac{\partial T}{\partial t}\Delta t\right)^2},\\ &= \left(\frac{1}{20}\right)\Delta t = \frac{0.2}{20} = 0.01\,\text{s}.\end{aligned}$$

The time period of the rigid pendulum can be quoted as,

$$T = (2.04 \pm 0.01)\,\text{s}. \qquad (1)$$

The number of swings N_t are related with the time period as,

$$T = \frac{t}{N_t}.$$

If we try with different values of swings N_t, we get the following results,

$$N_t = 79: \quad T = \frac{162.9}{79} = 2.062, \quad \text{and} \quad \Delta T = \frac{0.2}{79} = 0.002\,\text{s}.$$

$$N_t = 80: \quad T = \frac{162.9}{80} = 2.036, \quad \text{and} \quad \Delta T = \frac{0.2}{80} = 0.002\,\text{s}.$$

$$N_t = 81: \quad T = \frac{162.9}{81} = 2.011, \quad \text{and} \quad \Delta T = \frac{0.2}{81} = 0.002\,\text{s}.$$

Hence, we conclude that the time period for $N_t = 80$ is consistent with the time period of the rigid pendulum (1) and given as,

$$T = (2.036 \pm 0.002)\,s,$$

Taking $N_t = 80$ yields a more precise value of the time period T which also serves to fix the value of N_t for the next measurement.

2. A meter stick can be read to the nearest millimeter; a traveling microscope can be read to the nearest 0.1 mm. Suppose we want to measure the length of 2 cm with a precision of 1%. Choose the best answer. (3 points)

(a) The microscope has a precision of 0.5% while meter stick is precise upto 5%. The microscope is suited for length measurement.

(b) The meter stick has a precision of 0.25% while the microscope is precise upto 2.5%. The meter stick is suited for length measurement.

(c) The microscope has a precision of 0.25% while meter stick is precise upto 2.5%. The microscope is suited for length measurement.

(d) The meter stick has a precision of 1% while the microscope is precise upto 0.1%. The microscope is suited for length measurement.

(e) Both (a) and (c).

Solution:

The correct answer is (**c**).

Given length which we need to measure is,

$$l \ = \ 20\,\text{cm} = 20\,\text{mm}.$$

Since the given length is precise upto 1%, therefore,

$$1\%\,\text{precision of length}\,l \ = \ \frac{1}{100} \times 20 = 0.2\,\text{mm},$$

and we can quote the length as,

$$l \ = \ (20 \pm 0.2)\,\text{mm}.$$

Meter stick:

For the meter stick, the least count or the resolution is given as,

$$\Delta l_{\text{Meter stick}} \ = \ 1\,\text{mm}.$$

Since we prefer to take half of the length of the interval of the probability distribution function associated with the device. Likewise, this will be half of the least count of the device,

$$\Delta l_{\text{Meter stick}} \ = \ \left(\frac{\text{length of the interval}}{2}\right) = \left(\frac{1}{2}\right)\text{mm} = 0.5\,\text{mm}.$$

The fractional uncertainty is,

$$\begin{aligned}\frac{\Delta l_{\text{Meter stick}}}{l_{\text{Meter stick}}} \ &= \ \frac{0.5}{20} = 0.025\,\text{mm},\\ &= \ 0.025 \times 100 = 2.5\%\end{aligned}$$

Microscope:

Likewise, for microscope, the least count is,

$$\Delta l_{\text{Microscope}} \ = \ 0.1\,\text{mm},$$

and,

$$\Delta l_{\text{Microscope}} \ = \ \left(\frac{\text{length of the interval}}{2}\right) = \left(\frac{0.1}{2}\right)\text{mm} = 0.05\,\text{mm}.$$

The fractional uncertainty is,

$$\frac{\Delta l_{\text{Microscope}}}{l_{\text{Microscope}}} = \frac{0.05}{20} = 0.0025\,\text{mm},$$
$$= 0.0025 \times 100 = 0.25\%$$

3. My calculator returns the value of a measurand $x = 6.1234$, but I know that the x has a fractional uncertainty of 2%. How would I quote the final result? (3 points)

(a) $x = (6.12 \pm 0.12)$

(b) $x = (6.1 \pm 0.1)$

(c) $x = (6.12 \pm 0.31)$

(d) $x = (6.123 \pm 0.231)$

(e) $x = (6.1 \pm 0.5)$

Solution:

The correct answer is (**b**).

Given value of the measurand x is,

$$x = 6.1234,$$

and the fractional uncertainty is given as,

$$\frac{\Delta x}{x} = 2\%,$$

implying,

$$\Delta x = 2\% \times x = \left(\frac{2}{100}\right) \times 6.1234,$$
$$= 0.1224$$

Hence, the final result can be quoted as,

$$x = (6.12 \pm 0.1)$$

This result has 1 significant figure in the uncertainty value and the decimal places of both the original value and the uncertainty are at the same position. This is the proper way of quoting a measurement.

4. Three students measure the same resistance several times and their final answers are,

$$\begin{aligned} \text{Student 1}: \quad R &= (11 \pm 1)\,\Omega. \\ \text{Student 2}: \quad R &= (12 \pm 1)\,\Omega. \\ \text{Student 3}: \quad R &= (10 \pm 3)\,\Omega. \end{aligned}$$

If they decide to combine their results, the respective weights and the best estimate of R through weighted average would be, (3 points)

(a) Weights= $1, 1, 1/3$, Weighted average= $13.1\,\Omega$.

(b) Weights= $1, 1, 1/9$, Weighted average= $14.3\,\Omega$.

(c) Weights= $1, 1, 1/3$, Weighted average= $11.4\,\Omega$.

(d) Weights= $1, 1, 1/9$, Weighted average= $11.4\,\Omega$.

(e) None of the above.

Solution:

The correct answer is **(d)**.

The weight w_i of each measurement is the reciprocal square of the corresponding uncertainty σ_i, that is,

$$w_i = \frac{1}{\sigma_i^2}.$$

For the given measurands, the weights can be calculated as,

$$\begin{aligned} \text{Student 1}: \quad w_1 &= \left(\frac{1}{1^2}\right) = 1, \\ \text{Student 2}: \quad w_2 &= \left(\frac{1}{1^2}\right) = 1, \\ \text{Student 3}: \quad w_3 &= \left(\frac{1}{3^2}\right) = \frac{1}{9}. \end{aligned}$$

When a measurement is repeated many times, the best estimated value can be found out by the method of weighted average. The mathematical expression is given as,

$$\begin{aligned} R_{avg} &= \frac{\sum_{i=1}^{3} w_i R_i}{\sum_{i=1}^{3} w_i}, \\ &= \frac{(1 \times 11) + (1 \times 12) + (\frac{1}{9} \times 10)}{1 + 1 + \frac{1}{9}} = 11.42\,\Omega \end{aligned}$$

5. A student measures the area of a rectangle several times and concludes the standard uncertainty of the measurements is $\sigma_{\mathrm{A}} = 12\,\mathrm{cm}^2$. If all the uncertainties are truly random then the desired precision can be obtained by making enough measurements and averaging. How many measurements are needed to get a final uncertainty of $\pm 3\,\mathrm{cm}^2$. (3 points)

   (a) $N = 14$.

   (b) $N = 16$.

   (c) $N = 18$.

   (d) $N = 12$.

   (e) None of the above.

   Solution:

   The correct answer is **(b)**.

   The standard uncertainty of the measured data is $\sigma_{\mathrm{A}} = 12\,\mathrm{cm}^2$.
   The standard uncertainty in the mean value (σ_m) can be found out using the following relationship,

   $$\sigma_m = \frac{\sigma}{\sqrt{n}}, \tag{2}$$

   where σ is the standard uncertainty and n is the number of measurements taken.
   Rearranging the above expression yields,

   $$\begin{aligned} n &= \left(\frac{\sigma}{\sigma_m}\right)^2, \\ &= \left(\frac{12}{3}\right)^2 = 16. \end{aligned}$$

   Hence we conclude that we need to repeat the measurements 16 times to minimize the final uncertainty upto $\pm 3\,\mathrm{cm}^2$.

6. A semiconductor is sample of fairly pure silicon, and a simple theory suggests that the resistance R depends on the thermodynamic temperature T according to the relation,

   $$R = R_o \exp(E_g/2k_BT)$$

where R_o is the room temperature resistance, k_B is the Boltzmann constant and E_g is the band gap. The value of the band gap can be most easily found out by plotting, (3 points)

(a) Log-log plot.

(b) Semi-log plot.

(c) R versus E_g plot.

(d) R versus T plot.

(e) None of the above.

Solution:

The correct answer is **(b)**.

Given mathematical expression for the variation of resistance of a semiconductor with temperature is,

$$R = R_o \exp(E_g/2k_BT),$$

By taking log on both sides, we get,

$$\log(R) = \log(R_o) + \left(\frac{E_g}{2k_B}\right)\frac{1}{T}.$$

The above equation is a linear equation and a plot of $\log(R)$ versus $(1/T)$ will give a straight line whose slope is equal to $(E_g/2k_B)$. This is the concept of linearization that helps to get useful information from the data.

Semi-log plots are the graphs in which one axis is plotted on a logarithmic scale. These plots are generally used for exponential functions and cases in which one of the variables being plotted has a large range of data while other one has some restricted range.

7. The power P delivered to a resistance R by a current I is supposed to be given be the relation $P = I^2R$. To check this relation, a student sends several different currents through an unknown resistance immersed in a cup of water and measures the power delivered (by measuring the water's rise in temperature). The data is tabulated in Table (I).

Current I (A)	1.5	2.0	2.5	3.0	3.5	4.0
Power P (W)	270	380	620	830	1280	1600

TABLE I: Experimental data for current I and power dissipation P.

(a) Use mathematical expressions of least squares fitting of a straight line with equal weights to find the best estimated value of R. (8 points)

(b) Calculate uncertainty in R and quote your final result. Assume that I and P are measured with zero uncertainties. (4 points)

Solution:

(a) The relationship between current I and power dissipation P is,

$$P = I^2 R,$$

The power dissipation P term has a linear dependence on current squared I^2, hence we can take P as our dependent variable and I^2 as an independent variable. The slope of the above equation gives us the value of resistance R.

The mean values of the measurands are,

$$\begin{aligned}\bar{P} &= \frac{\sum_{i=1}^{6} P_i}{n} \\ &= \frac{270 + 380 + 620 + 830 + 1280 + 1600}{6} \\ &= 830\,\mathrm{W}.\end{aligned}$$

$$\begin{aligned}\bar{I^2} &= \frac{\sum_{i=1}^{6} I^2}{n} \\ &= \frac{2.25 + 4.00 + 6.25 + 9.00 + 12.25 + 16.00}{6} \\ &= 8.29\,\mathrm{A}^2.\end{aligned}$$

The best estimated value of the slope can be found out using the following expression,

$$m = \frac{\sum_{i=1}^{N} y_i (x_i - \bar{x})}{\sum_{i=1}^{N} (x_i - \bar{x})^2}.$$

However utilizing our dependent and independent variables, the above expression becomes,

$$m = \frac{\sum_{i=1}^{6} P_i(I_i^2 - \bar{I}^2)}{\sum_{i=1}^{6}(I_i^2 - \bar{I}^2)^2},$$

which implies,

$$\begin{aligned} m &= \frac{[270(2.25 - 8.29) + 380(4.00 - 8.29) + 620(6.25 - 8.29)}{[(2.25 - 8.29)^2 + (4.00 - 8.29)^2 + (6.25 - 8.29)^2}, \\ &= \frac{+830(9.00 - 8.29) + 1280(12.25 - 8.29) + 1600(16.00 - 8.29)]}{+(9.00 - 8.29)^2 + (12.25 - 8.29)^2 + (16.00 - 8.29)^2]}, \\ &= \frac{13468}{134.68}, \\ &= 100\,\Omega. \end{aligned}$$

(b) The intercept can be calculated as,

$$c = \bar{y} - m\bar{x}.$$

Hence,

$$\begin{aligned} c &= \bar{P} - m\bar{I}^2 = 830 - (100)(8.29), \\ &= 1\,\text{W}. \end{aligned}$$

Uncertainty in slope m is given as,

$$u_m = \sqrt{\frac{\sum_i^N d_i^2}{D(N-2)}},$$

where,

$$\begin{aligned} d_i &= y_i - mx_i - c, \\ D &= \sum_i^N (x_i - \bar{x})^2. \end{aligned}$$

Now,

$$\begin{aligned} u_m &= \sqrt{\frac{(44.13)^2 + (-20.77)^2 + (-5.64)^2 + (-70.48)^2 + (54.70)^2 + (-0.08)^2}{4 \times [(-6.04)^2 + (-4.29)^2 + (-2.04)^2 + (0.71)^2 + (3.96)^2 + (7.71)^2]}}, \\ &= 4.39\,\Omega. \end{aligned}$$

Hence, the final value of resistance can be quoted as,

$$R = (100 \pm 4)\,\Omega.$$