Assignment 5: Solution

1. For a particle incident on a potential step with $E < V_0$, show that the magnitudes of the amplitudes of the incident and reflected waves functions are the same. Find the phase shift that the wave function acquires on reflection.

Answer

For the given case of $E < V_0$, the time-independent Schrodinger equation solved in each region yields the wave functions:

$$\psi(x) = A e^{ik_1x} + B e^{-ik_1x} \quad x < 0$$
$$\psi(x) = Ce^{-\alpha x} \quad x > 0$$

where

$$k_1 = \sqrt{\frac{2mE}{\hbar^2}}, \quad \alpha = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}.$$ 

Note $\alpha > 0$. Applying the boundary conditions (continuity of the wave function and its first derivative at $x = 0$) yields

$$A + B = C$$
$$ik_1 A - ik_1 B = -\alpha C.$$

$A$ is the amplitude of the incident wave, $B$ is the amplitude of the reflected wave and $C$ is the amplitude of the transmitted wave. Solving these equations, obtain

$$B = \left(\frac{ik_1 + \alpha}{ik_1 - \alpha}\right) A \quad \text{and} \quad C = \frac{2ik_1}{(ik_1 - \alpha)} A.$$

This implies that

$$|B| = \left|\frac{ik_1 + \alpha}{ik_1 - \alpha}\right| |A|$$
$$= \left|\frac{ik_1 + \alpha}{ik_1 - \alpha}\right| |A|.$$
\[ \frac{B}{A} = e^{\tan^{-1}(k_1/\alpha)} = e^{\tan^{-1}(k_1/\alpha) - \tan^{-1}(-k_1/\alpha)} = e^{2\tan^{-1}(k_1/\alpha)}, \]

where we have used,

\[ \tan^{-1}(x) = y_1 \Rightarrow \tan y_1 = x \]
\[ \tan^{-1}(-x) = y_2 \Rightarrow \tan y_2 = -x \Rightarrow \tan y_2 = -\tan y_1. \]

Now since

\[ k_1 = \frac{2mE}{2m(V_0 - E)} = \frac{E}{V_0 - E}, \]

the phase shift acquired upon reflection is \( 2\tan^{-1} \sqrt{E/(V_0 - E)} \).

2. Use the Heisenberg indeterminacy relation to estimate the ground state energy of a particle in an infinite square well potential.

**Answer**

The uncertainty principle is \( \Delta x \Delta p \geq \hbar/2 \). If the particle is confined in the well \( \Delta x \sim L \) leading to \( \Delta p \geq \hbar/2L \). The momentum \( p \) has to be at least of the order of \( \Delta p \), hence \( p > \hbar/2L \) which means that the energy is at least \( p^2/2m = \hbar^2/8mL^2 \). The ground state energy is given by \( \pi^2\hbar^2/8mL^2 \), showing a qualitative agreement with the minimum energy predicted by Heisenberg’s uncertainty.

3. At \( t = 0 \) a particle in an infinite square well potential

\[ V(x) = \begin{cases} 0, & 0 < x < L, \\ \infty, & \text{elsewhere,} \end{cases} \]

is in a state described by the wave function

\[ \Psi(x, 0) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi}{L} x\right) + i \sqrt{\frac{1}{L}} \sin\left(\frac{2\pi}{L} x\right), & 0 < x < L, \\ 0, & \text{elsewhere,} \end{cases} \]
Determine,
(a) the probability \( P(E_n, t) \) that a measurement of the energy will yield the value \( E_n \).
(b) \( \langle E \rangle(t) \),  
(c) \( \langle x \rangle(t) \),  
(d) \( \langle p \rangle(t) \).

**Answer**

For a particle in an infinite square well potential described by normalized wave function
\[ |\psi(0)\rangle = c_2 |\psi_2\rangle + c_3 |\psi_3\rangle \]
\[ = \frac{1}{\sqrt{3}} |\psi_2\rangle + i \sqrt{\frac{2}{3}} |\psi_3\rangle. \]

At time \( t \), the state becomes
\[ |\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle = \frac{e^{-iE_2t/\hbar}}{\sqrt{3}} |\psi_2\rangle + i \sqrt{\frac{2}{3}} e^{-iE_3t/\hbar} |\psi_3\rangle, \]

Here \( E_n \) is the energy of the state \( |\psi_n\rangle \) which is an eigenstate of the Hamiltonian. These eigenstates are found by solving the time-independent Schrodinger equation inside the well
\[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi_n(x) = E_n \psi_n(x). \]

For an infinite well the wavefunction corresponding to the eigenstates of Hamiltonian
\[ \psi_n(x) = \langle n|\psi_n \rangle \]
\[ = \frac{\sqrt{2}}{L} \sin \left( \frac{n\pi x}{L} \right). \]

(a) The probability of measuring \( E_n \) at time \( t \) is
\[ P(E_n, t) = |\langle \psi_n | \psi(t) \rangle|^2 \]
Now
\[ \langle \psi_n | \psi(t) \rangle = \frac{e^{-iE_2t/\hbar}}{\sqrt{3}} \langle \psi_n | \psi_2 \rangle + i \sqrt{\frac{2}{3}} e^{-iE_3t/\hbar} \langle \psi_n | \psi_3 \rangle \]
\[ = \frac{e^{-iE_2t/\hbar}}{\sqrt{3}} \delta_{2n} + i \sqrt{\frac{2}{3}} e^{-iE_3t/\hbar} \delta_{3n}. \]
Here $\delta_{jk}$ is a Kronecker delta (equal to 1 when $j = k$ and 0 otherwise). Therefore,

$$P(E_n, t) = |\langle \psi_n | \psi(t) \rangle|^2$$

$$= \left( \frac{e^{iE_2t/\hbar}}{\sqrt{3}} \delta_{2n} - i \sqrt{\frac{2}{3}} e^{iE_3t/\hbar} \delta_{3n} \right) \left( \frac{e^{-iE_2t/\hbar}}{\sqrt{3}} \delta_{2n} + i \sqrt{\frac{2}{3}} e^{-iE_3t/\hbar} \delta_{3n} \right)$$

$$= \frac{1}{3} \delta_{2n} + \frac{2}{3} \delta_{3n}. \quad \text{(Since } \delta_{2n} \delta_{3n} = \delta_{3n} \delta_{2n} = 0)$$

Hence the probability of measuring the energy $E_2$ at time $t$ is $1/3$ and probability of measuring the energy $E_3$ at time $t$ is $2/3$. One never measures the energy $E_k$ with $k \neq 2, 3$.

(b) To find the expectation value of $E$, first we need to determine $\Psi(x, t)$;

$$\Psi(x, t) = \sum_n c_n \psi_n(x) e^{-iE_nt/\hbar} \quad \text{(3)}$$

$$= \frac{1}{\sqrt{3}} \psi_2(x) e^{-iE_2t/\hbar} + i \sqrt{\frac{2}{3}} \psi_3(x) e^{-iE_3t/\hbar}.$$

I hope you would understand how Eq.(3) is obtained. Given here is first a derivation of Eq.(3).

The Schrodinger equation is solved through

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar} |\psi(0)\rangle.$$

In the position representation,

$$\Psi(x, t) = \langle x | \psi(t) \rangle = \langle x | e^{-i\hat{H}t/\hbar} |\psi(0)\rangle \quad \text{(4)}.$$

The initial state $|\psi(0)\rangle$ can be written as a superposition of the eigenstates of the Hamiltonian which form a complete basis,

$$|\psi(0)\rangle = \sum_n c_n(0) |\psi_n\rangle,$$

whereupon inserting the above into Eq (4) results in

$$\Psi(x, t) = \langle x | e^{-i\hat{H}t/\hbar} \left( \sum_n c_n(0) |\psi_n\rangle \right)$$

$$= \sum_n c_n(0) e^{-iE_nt/\hbar} \psi_n(x).$$
which is the desired form Eq (3).

Now, we present three methods for calculating $\langle E \rangle$. First, we have

$$\langle E \rangle = \langle \Psi(x,t) | \hat{H} | \Psi(x,t) \rangle$$

$$= \int_0^L dx \Psi^\dagger(x,t) \hat{H} \Psi(x,t)$$

$$= \int_0^L dx \left( \frac{1}{\sqrt{3}} \psi_2^\dagger(x) e^{iE_2t/\hbar} - i \sqrt{\frac{2}{3}} \psi_3^\dagger(x) e^{iE_3t/\hbar} \right) \hat{H} \left( \frac{1}{\sqrt{3}} \psi_2(x) e^{-iE_2t/\hbar} + i \sqrt{\frac{2}{3}} \psi_3(x) e^{-iE_3t/\hbar} \right)$$

$$= \frac{1}{3} \int_0^L dx \left( \psi_2^\dagger(x) e^{iE_2t/\hbar} - i \sqrt{2} \psi_3^\dagger(x) e^{iE_3t/\hbar} \right) \left( E_2 \psi_2(x) e^{-iE_2t/\hbar} + i \sqrt{2} E_3 \psi_3(x) e^{-iE_3t/\hbar} \right)$$

$$= \frac{1}{3} \left[ E_2 \int_0^L |\psi_2(x)|^2 dx + 2E_3 \int_0^L |\psi_3(x)|^2 dx + 0 \right]$$

where

$$\int_0^L \psi_2(x) \psi_3(x) dx = \frac{2}{L} \int_0^L \sin \left( \frac{2\pi x}{L} \right) \sin \left( \frac{3\pi x}{L} \right) dx = 0.$$

So the expectation value of $E$ is

$$\langle E \rangle = \frac{1}{3} \left[ E_2(1) + 2E_3(1) \right] = \frac{E_2}{3} + \frac{2E_3}{3}.$$

Now the second method. Since the energy is quantized in discrete steps, one can find its average value through the discrete sum given below and use probabilities determined from the previous part;

$$\langle E \rangle = \sum_i P(E_i) E_i$$

$$= P(E_2) E_2 + P(E_3) E_3 = \frac{1}{3} E_2 + \frac{2}{3} E_2.$$

In yet a third approach, one can solve this part by directly performing the calculation in the Hilbert space as seen below:

$$|\psi(t)\rangle = \sum_n c_n(0) e^{-i\hat{H}t/\hbar} |\psi_n\rangle$$

$$\langle E \rangle = \langle \psi(t) | \hat{H} | \psi(t) \rangle = \text{to be found}$$

$$\hat{H} |\psi(t)\rangle = \sum_n c_n(0) \hat{H} e^{-iE_n t/\hbar} |\psi_n\rangle$$

$$= \sum_n c_n(0) E_n e^{-iE_n t/\hbar} |\psi_n\rangle$$

Due Date: April. 28, 2014, 10:00 am
\[ \langle \psi(t) | \hat{H} | \psi(t) \rangle = \left( \sum_m c_m(0) \langle \psi_m | e^{i\hat{H}t/\hbar} \right) \left( \sum_n c_n(0) E_n e^{-iE_nt/\hbar} \langle \psi_n | \right) \]
\[ = \sum_m \sum_n c_m(0) c_n(0) E_n e^{i(E_m - E_n)t/\hbar} \langle \psi_m | \psi_n \rangle \]
\[ = \sum_m \sum_n c_m(0) c_n(0) E_n e^{i(E_m - E_n)t/\hbar} \delta_{mn} \]
\[ = \sum_n |c_n(0)|^2 E_n = \frac{E_2}{3} + \frac{2}{3} E_3 \quad \text{(since } c_n(0) = 0 \text{ when } n \neq 2, 3) \]

**Appendical discussion**

Here we digress motivating the student towards Eq.(5). From the discussion on page 224 and 225 of Beck’s book, you will recognize that,

\[ \langle x | \hat{p}_x | \psi \rangle = \langle x \left| -i\hbar \frac{\partial}{\partial x} \right| \psi \rangle = -i\hbar \frac{\partial}{\partial x} \langle x | \psi \rangle = -i\hbar \frac{\partial}{\partial x} \psi(x) \]

Furthermore, it is straightforward to realize that,

\[ \langle x | \hat{x} | \psi \rangle = x \psi(x) \]

Therefore, the expectation value of momentum can be found in the wave mechanical formulation as,

\[ \langle \hat{p}_x \rangle = \langle \psi | \hat{p}_x | \psi \rangle = \int_x dx \langle \psi | x \rangle \langle x | \hat{p}_x | \psi \rangle \]
\[ = \int_x dx \psi^\dagger(x) \left(-i\hbar \frac{\partial}{\partial x}\right) \psi(x). \]

Similarly, we have

\[ \langle \hat{p}_x^2 \rangle = \int_x dx \psi^\dagger(x) \left(-i\hbar \frac{\partial}{\partial x}\right)^2 \psi(x) \quad \text{and so on.} \]

The expectation value of position is

\[ \langle x \rangle = \langle \psi | \hat{x} | \psi \rangle = \int_x dx \langle \psi | x \rangle \langle x | \hat{x} | \psi \rangle \]
\[ = \int_x dx \psi^\dagger(x) x \psi(x). \]

Now the energy \( E = \frac{p^2}{2m} + V(x) \),

\[ \langle E \rangle = \left\langle \left( \frac{p^2}{2m} + V(x) \right) \right\rangle = \int_x dx \psi^\dagger(x) \left( \frac{(-i\hbar \partial/\partial x)^2}{2m} + V(x) \right) \psi(x) \]
\[ = \int_x dx \psi^\dagger(x) \hat{H} \psi(x), \]
which is Eq (5). This is how you perform expectation value measurements in
the wavefunction representation.

\[
\langle x \rangle = \langle \Psi(x,t) | \hat{x} | \Psi(x,t) \rangle
\]

\[
= \int_0^L dx \Psi^\dagger(x,t)x \Psi(x,t)
\]

\[
= \int_0^L dx |\Psi(x,t)|^2 x
\]

(c)

\[
\langle x \rangle = \langle \Psi(x,t) | \hat{x} | \Psi(x,t) \rangle
\]

\[
= \int_0^L dx \Psi^\dagger(x,t)x \Psi(x,t)
\]

\[
= \int_0^L dx |\Psi(x,t)|^2 x
\]

Now from Eq (3),

\[
\Psi(x,t) = \frac{1}{\sqrt{3}} \psi_2(x)e^{-iE_2 t/\hbar} + i \sqrt{\frac{2}{3}} \psi_3(x)e^{-iE_3 t/\hbar}
\]

\[
|\Psi(x,t)|^2 = \left( \frac{1}{\sqrt{3}} \psi_2^\dagger(x)e^{iE_2 t/\hbar} - i \sqrt{\frac{2}{3}} \psi_3^\dagger(x)e^{iE_3 t/\hbar} \right) \left( \frac{1}{\sqrt{3}} \psi_2(x)e^{-iE_2 t/\hbar} + i \sqrt{\frac{2}{3}} \psi_3(x)e^{-iE_3 t/\hbar} \right)
\]

\[
= \frac{1}{3} |\psi_2(x)|^2 + \frac{2}{3} |\psi_3(x)|^2 + i \frac{\sqrt{2}}{3} \psi_2^\dagger(x)\psi_3(x)e^{i(E_2-E_3)t/\hbar} - i \frac{\sqrt{2}}{3} \psi_3^\dagger(x)\psi_2(x)e^{-i(E_2-E_3)t/\hbar}
\]

Since \(\psi_2^\dagger(x) = \psi_2(x) = \sqrt{\frac{2}{L}} \sin \left( \frac{2\pi x}{L} \right)\) and \(\psi_3^\dagger(x) = \psi_3(x) = \sqrt{\frac{2}{L}} \sin \left( \frac{3\pi x}{L} \right)\).

\[
\psi_2^\dagger(x)\psi_3(x) = \psi_3^\dagger(x)\psi_2(x)
\]

\[
= \frac{2}{L} \sin \left( \frac{2\pi x}{L} \right) \sin \left( \frac{3\pi x}{L} \right)
\]

\[
= \frac{1}{L} \left[ \cos \left( \frac{\pi x}{L} \right) - \cos \left( \frac{5\pi x}{L} \right) \right]
\]

\[
|\Psi(x,t)|^2 = \frac{1}{3} |\psi_2(x)|^2 + \frac{2}{3} |\psi_3(x)|^2 + i \frac{\sqrt{2}}{3} \frac{1}{L} \left[ \cos \left( \frac{\pi x}{L} \right) - \cos \left( \frac{5\pi x}{L} \right) \right] (2i) \sin \left( \frac{(E_2-E_3)t}{\hbar} \right).
\]

Inserting this expression into Eq (6), we can evaluate as given below. The first
integral is

\[
\int_0^L x |\psi_2(x)|^2 dx = \frac{2}{L} \int_0^L x \sin^2 \left( \frac{2\pi x}{L} \right) = \frac{2}{L} \left( \frac{L^2}{4} \right) = \frac{L}{2},
\]

and the second integral is also,

\[
\int_0^L x |\psi_3(x)|^2 dx = \frac{L}{2},
\]

Likewise the third integral becomes

\[
\int_0^L x \left[ \cos \left( \frac{\pi x}{L} \right) - \cos \left( \frac{5\pi x}{L} \right) \right] dx = -\frac{2L^2}{\pi^2} + \frac{2L^2}{25\pi^2} = -\frac{48L^2}{25\pi^2}.
\]
Hence the expectation value is

\[
\langle x \rangle = \frac{1}{3} \left[ \frac{3}{2} L + 2 \sqrt{2} \frac{48 L}{25 \pi^2} \sin \left( \frac{(E_2 - E_3)t}{\hbar} \right) \right] = \frac{L}{2} + \frac{32 \sqrt{2} L}{25 \pi^2} \sin \left( \frac{(E_2 - E_3)t}{\hbar} \right).
\]

Note that \( \langle x \rangle \) depends on time.

(d)

\[
\langle p \rangle = \langle \Psi(x,t) | \hat{p} | \Psi(x,t) \rangle = \int_0^L dx \psi_1(x,t) \left( -i\hbar \frac{\partial}{\partial x} \right) \psi(x,t) = \frac{-i\hbar}{3} \int_0^L dx \left( \psi_1(x)e^{iE_2t/\hbar} - i\sqrt{2}\psi_3(x)e^{iE_3t/\hbar} \right) \left( \frac{\partial \psi_2(x)}{\partial x} e^{-iE_2t/\hbar} + i\sqrt{2} \frac{\partial \psi_3(x)}{\partial x} e^{-iE_3t/\hbar} \right).
\]

Here both terms in the first integral vanishes,

\[
\therefore \int_0^L dx \psi_1(x) \frac{\partial \psi_2(x)}{\partial x} = 4\pi \int_0^L \sin \left( \frac{2\pi x}{L} \right) \cos \left( \frac{2\pi x}{L} \right) dx = 0,
\]

and

\[
\int_0^L dx \psi_3(x) \frac{\partial \psi_2(x)}{\partial x} = \frac{6\pi}{L^2} \int_0^L \sin \left( \frac{2\pi x}{L} \right) \cos \left( \frac{3\pi x}{L} \right) dx = \frac{6\pi}{L^2} \left( \frac{-4L}{5\pi} \right) = \frac{-24}{5L},
\]

\[
\int_0^L dx \psi_3(x) \frac{\partial \psi_2(x)}{\partial x} = \frac{24}{5L}.
\]

Therefore,

\[
\langle p \rangle = \frac{i\hbar}{3} (i\sqrt{2}) \left( \frac{24}{5L} \right) \left[ e^{i(E_3-E_2)t/\hbar} + e^{-i(E_3-E_2)t/\hbar} \right] = -\frac{\hbar 16 \sqrt{2}}{5L} \cos \left( \frac{(E_3 - E_2)t}{\hbar} \right).
\]

4. At \( t = 0 \) a particle in an infinite square well potential [Eq (1)] is in a state described by the wave function

\[
\Psi(x,0) = \begin{cases} \sqrt{\frac{30}{L^2}}x(L-x), & 0 < x < L, \\ 0 & \text{elsewhere}. \end{cases}
\]
Determine
(a) $\Psi(x, t)$,
(b) the probability $P(E_n, t)$ that a measurement of the energy at time $t$ will yield the value $E_n$,
(c) $\langle E \rangle(t)$,  (d) $\langle x \rangle(t)$,  (e) $\langle p \rangle(t)$.

**Answer**

Let's first verify if $\Psi(x, 0)$ is normalized.

\[
|\Psi(x, 0)|^2 = \frac{30}{L^5} x^2(L - x)^2 \quad x \in [0, L]
\]

\[
= \frac{30}{L^5} x^2(L^2 + x^2 - 2Lx)
\]

\[
= \frac{30}{L^5} (L^2x^2 + x^4 - 2Lx^3)
\]

\[
\int_0^L dx |\Psi(x, 0)|^2 = \frac{30}{L^5} \int_0^L dx (L^2x^2 + x^4 - 2Lx^3)
\]

\[
= \frac{30}{L^5} \left[ \frac{L^2x^3}{3} + \frac{x^5}{5} - \frac{2Lx^4}{4} \right]_0^L
\]

\[
= \frac{30}{L^5} \left( \frac{L^5}{3} + \frac{L^5}{5} - \frac{2L^5}{4} \right) = 30 \left( \frac{20 + 12 - 30}{60} \right) = 1.
\]

Hence $\Psi(x, 0) = \sqrt{\frac{30}{L^5}} x(L - x)$ is properly normalized.

(a) In Q2, we have already seen that

\[
\Psi(x, t) = \sum_n c_n(0) e^{-iE_n t/\hbar} \psi_n,
\]
where $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$ are the eigenstates of the Hamiltonian and

$$c_n(0) = \langle \psi_n | \Psi(x,0) \rangle = \int_{x=0}^{L} dx \langle \psi_n | x \rangle \langle x | \Psi(x,0) \rangle$$

$$= \int_{0}^{L} dx \psi_n^\dagger(x) \Psi(x,0)$$

$$= \int_{0}^{L} dx \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \sqrt{\frac{30}{L^5}} x(L-x)$$

$$= \sqrt{\frac{60}{L^5}} \int_{0}^{L} dx \sin\left(\frac{n\pi x}{L}\right) x(L-x).$$

You can leave this integral as it is, call it some $I$, or solve to obtain

$$I_n = \int_{0}^{L} dx \sin\left(\frac{n\pi x}{L}\right) x(L-x) = \begin{cases} 0, & \text{for } n \text{ even} \\ \frac{4L^3}{3\pi^3}, & \text{for } n \text{ odd} \end{cases} \quad (7)$$

Hence

$$\Psi(x,t) = \sqrt{\frac{2}{L}} \sqrt{\frac{60}{L^5}} \sum_{n=1}^{\infty} I_n e^{-iE_n t/\hbar} \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sqrt{\frac{120}{L^7}} \sum_{n} I_n e^{-iE_n t/\hbar} \sin\left(\frac{n\pi x}{L}\right) = \sqrt{\frac{120}{L^7}} \sum_{n=1,3,5,\ldots} \frac{4L^3}{n^3\pi^3} e^{-iE_n t/\hbar} \sin\left(\frac{n\pi x}{L}\right).$$

Note that the initial state $\Psi(x,0)$ is a superposition of all the eigenstates, and is not a stationary state of the Hamiltonian.

(b) The probability is $|\langle \psi_1 | \Psi(x,t) \rangle|^2$

Now $\langle \psi_1 | \Psi(x,t) \rangle = \langle \psi_1 | e^{-i\hat{H}t/\hbar} | \Psi(x,0) \rangle$

$$= \langle \psi_1 | e^{-i\hat{H}t/\hbar} \left( \sum_{n=1}^{\infty} c_n(0) | \psi_n \rangle \right)$$

$$= \sum_{n=1}^{\infty} c_n(0) \langle \psi_1 | e^{-i\hat{H}t/\hbar} | \psi_n \rangle$$

$$= \sum_{n=1}^{\infty} c_n(0) \langle \psi_1 | \psi_n \rangle e^{-iE_n t/\hbar}$$

$$= c_1(0) e^{-iE_1 t/\hbar}$$

$$|\langle \psi_1 | \Psi(x,t) \rangle|^2 = |c_1(0)|^2 = \left( \frac{4\sqrt{60}}{\pi^3} \right)^2 \approx 0.999.$$
(c) \[
\langle E \rangle(t) = \langle \Psi(x,t) | \hat{H} | \Psi(x,t) \rangle = \langle \Psi(x,t) | e^{-i\hat{H}t/\hbar} | \Psi(x,t) \rangle = \langle \Psi(x,0) | \hat{H} | \Psi(x,0) \rangle.
\]
Since \( |\Psi(x,0)\rangle = \sum_n c_n(0) |\psi_n \rangle \),
\[
\langle E \rangle(t) = \left( \sum_m c_m^\dagger(0) |\psi_m \rangle \right) \hat{H} \left( \sum_n c_n(0) |\psi_n \rangle \right) = \sum_m c_m^\dagger(0) c_n(0) E_n \langle \psi_m | \psi_n \rangle = \sum_{n=0}^\infty |c_n(0)|^2 E_n
\]
\[
= \sum_{n=1,3,5,\ldots}^\infty \frac{E_n 16(60)}{n^6 \pi^6} = \sum_{n=1,3,5,\ldots}^\infty \frac{n^2 \hbar^2 \pi^2}{2mL^2} \frac{16(60)}{n^6 \pi^6}
\]
\[
\langle E \rangle(t) = \sum_{n=1,3,5,\ldots}^\infty \frac{480 \hbar^2}{n^4 \pi^4 mL^2}.
\]

The expectation value of energy is independent of time.

(d) The easiest way to find the expectation value of \( x \) is through wavefunction,
\[
\langle x \rangle = \langle \Psi(x,t) | \hat{x} | \Psi(x,t) \rangle = \int_0^L dx \Psi^\dagger(x,t) x \Psi(x,t).
\]
Since \( \Psi(x,t) = \sqrt{\frac{120}{L^7}} \sum_n I_n e^{-iE_n t/\hbar} \sin\left(\frac{n\pi x}{L}\right) \), where \( I_n \) is given in Eq. (7) and only exists for \( n \) odd.
\[
\therefore \langle x \rangle = \left( \frac{120}{L^7} \right) \int_0^L dx \left( \sum_m I_m e^{iE_m t/\hbar} \sin\left(\frac{m\pi x}{L}\right) \right) x \left( \sum_n I_n e^{-iE_n t/\hbar} \sin\left(\frac{n\pi x}{L}\right) \right)
\]
\[
= \left( \frac{120}{L^7} \right) \sum_m \sum_n I_m I_n e^{i(E_m - E_n)t/\hbar} \int_0^L dx \left( x \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) \right).
\]
Now
\[
\int_0^L dx x \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) = \frac{L^2}{2\pi^2} \left[ -\frac{1}{(m-n)^2} + \frac{1}{(m+n)^2} \right] + \frac{\cos(m-n)\pi + (m-n)\pi \sin(m-n)\pi}{(m-n)^2} - \frac{\cos(m+n)\pi + (m+n)\pi \sin(m+n)\pi}{(m+n)^2}.
\]
\[
\begin{align*}
\int_e^L dx \sin \left( \frac{m\pi x}{L} \right) \cos \left( \frac{n\pi x}{L} \right) &= \frac{1}{2} \left[ \int_0^L dx \sin \left( \frac{(m+n)\pi x}{L} \right) + \sin \left( \frac{(m-n)\pi x}{L} \right) \right] \\
&= \frac{1}{2} \left[ \left. -\cos \left( \frac{(m+n)\pi x}{L} \right) \right|_0^L + \cos \left( \frac{(m-n)\pi x}{L} \right) \right|_0^L \\
&= \frac{-L}{2\pi} \left[ \frac{\cos \left( (m+n)\pi - 1 \right)}{(m+n)} + \frac{\cos \left( (m-n)\pi - 1 \right)}{(m-n)} \right] \\
&= \frac{-L}{2\pi} \left[ \frac{2m \cos (m\pi) \cos (n\pi) - 2m}{(m^2 - n^2)} \right] \\
&= \frac{-Lm}{\pi} \left[ \frac{\cos (m\pi) \cos (n\pi) - 1}{(m^2 - n^2)} \right].
\end{align*}
\]

\[
\therefore \langle p \rangle = \left( \frac{i120\hbar}{L^7} \right) \sum_m \sum_n I_m I_n e^{i(E_m - E_n)t/\hbar} mn \left[ \frac{\cos (m\pi) \cos (n\pi) - 1}{(m^2 - n^2)} \right].
\]

For \( n = m, \langle p \rangle = \infty. \) In all other cases \( \langle p \rangle = 0. \)
5. An electron is inside a nanowire, confined to one dimension. The length of the wire is \( L \). The quantum state of the electron is described by a particle in an infinite well. A measurement of the energy of the electron yields \( \frac{\hbar^2 \pi^2}{2mL^2} \), indicating that the electron is in the ground state, \( |\psi_L(x)\rangle \). Suddenly, a mechanical force is applied to the nanowire doubling its length. The new potential well at the instant of stretching \( (t = 0) \) is shown in Fig (b). In our notation \( \psi_n \) refers to the \( n \)'th eigenstate in a well of length \( L \), and \( \psi_{2L}^n \) refers to the \( n \)'th eigenstate in a well of length \( 2L \). The ground state has \( n = 1 \). Similar notation is used for energies.

(a) Is \( |\psi_L^1(x)\rangle \) is a stationary state of the new configuration?

(b) Write \( |\psi_L^1(x)\rangle \) in the position basis, i.e. find \( \psi_1(x) = \langle x|\psi_L^1(x)\rangle \).

(c) What is the ground state energy of the particle in the well of length \( 2L \)? Call it \( E_{2L}^1 \). What is the ground state wavefunction \( \psi_{2L}^1(x) \)?

(d) Now the energy is measured at \( t = 0^+ \), right after the well expands. What is the probability of measuring the energy to be \( E_{2L}^1 \)?

(e) What is the probability of measuring the energy to be \( E_{2L}^2 = 4E_{2L}^1 \) and \( E_{2L}^3 = 9E_{2L}^1 \)?

(f) Sketch the wavefunction \( \psi_{1L}^1 \) and \( \psi_{2L}^1, \psi_{2L}^2, \psi_{3L}^2 \) inside the well of length \( 2L \). Predict if you would have some probability of measuring the energy \( E_{2L}^n \) when \( n \) is even, or odd? Justify your answer.

(g) Calculate the time dependent wavefunction \( \Psi(x,t) = \langle x|\psi(t)\rangle \) in the well of length \( 2L \).
**Answer**

The initial state $|\psi_L^I(x,0)\rangle$ is not an eigenstate of the expanded well. So one needs to write this as a superposition of the eigenstates of the new well, $|\psi_n^{2L}\rangle$.

Let $|\psi_L^I(x,0)\rangle = \sum_{n=1}^{\infty} c_n(0)|\psi_n^{2L}\rangle$,

where $c_n(0) = \langle \psi_n^{2L} | \psi_L^I(x,0) \rangle$

and $|\psi_n^{2L}\rangle = \sqrt{\frac{2}{2L}} \sin\left(\frac{n\pi x}{2L}\right) = \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi x}{2L}\right)$.

Therefore,

$$c_n(0) = \int_{x=0}^{2L} dx \langle \psi_n^{2L} | x \rangle \langle x | \psi_L^I(x,0) \rangle$$

$$= \int_{x=0}^{2L} dx \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi x}{2L}\right) \Psi_1^I(x,0)$$

$$= \int_{x=0}^{L} dx \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi x}{2L}\right) \sqrt{\frac{1}{L}} \sin\left(\frac{\pi x}{L}\right).$$

Note that we have changed the limits of integration from $[0,2L]$ to $[0,L]$ because $\Psi_1^I(x,0) = 0$ in $x \in [L/2,L]$. Proceeding further,

$$c_n(0) = \frac{1}{L} \int_{x=0}^{L} dx \sin\left(\frac{n\pi x}{2L}\right) \sin\left(\frac{\pi x}{L}\right)$$

$$= \frac{1}{2L} \int_{x=0}^{L} dx \left[ \cos\left(\frac{n}{2} - \frac{1}{2}\right) \pi x - \cos\left(\frac{n}{2} + 1\right) \pi x \right]$$

$$= \frac{1}{2L} \left[ \sin\left(\frac{n}{2} - \frac{1}{2}\right) \pi \right]_0^L - \sin\left(\frac{n}{2} + 1\right) \pi \right]_0^L$$

$$= \frac{1}{2\pi} \left[ \sin\left(\frac{n}{2} - \frac{1}{2}\right) - \sin\left(\frac{n}{2} + 1\right) \right]$$

Now

$$\sin\left(\frac{n}{2} \pm 1\right) = \sin\left(\frac{n\pi}{2} \pm \pi\right)$$

$$= \sin\left(\frac{n\pi}{2}\right) \cos\pi \pm \cos\left(\frac{n\pi}{2}\right) \sin\pi = -\sin\left(\frac{n\pi}{2}\right)$$

$\therefore c_n(0) = -\frac{1}{2\pi} \sin\left(\frac{n\pi}{2}\right) \left( \frac{1}{n-2} - \frac{1}{n+1} \right)$

$$= -\frac{1}{2\pi} \sin\left(\frac{n\pi}{2}\right) \left( \frac{2}{n-2} - \frac{2}{n+2} \right)$$

**Due Date: April 28, 2014, 10:00 am**
\[ \frac{1}{\pi} \sin \left( \frac{n\pi}{2} \right) \left( \frac{n + 2 - (n - 2)}{n^2 - 4} \right) \]

\[ = \frac{1}{\pi(n^2 - 4)} \sin \left( \frac{n\pi}{2} \right) (-4) \]

\[ = \frac{4}{\pi(4 - n^2)} \sin \left( \frac{n\pi}{2} \right) \begin{cases} 
\frac{4}{\pi(4 - n^2)}, & n = 1, 5, 9, \\
\frac{-4}{\pi(4 - n^2)}, & n = 3, 7, 11, \\
0, & n \text{ even.}
\end{cases} \]

With this background we can attempt the solution.

(a) No it is not. The stationary state of the new configuration are

\[ \langle x | \psi_{2L}^L (x) \rangle = \sqrt{\frac{2}{2L}} \sin \left( \frac{n\pi x}{2L} \right) = \sqrt{\frac{1}{L}} \sin \left( \frac{n\pi x}{2L} \right) \]

(b)

\[ \langle x | \psi_1^L (x) \rangle = \sqrt{\frac{2}{L}} \sin \left( \frac{\pi x}{L} \right). \]

(c)

\[ E_{1L}^2 = \frac{\pi^2 \hbar^2}{2m(2L)^2} = \frac{\pi^2 \hbar^2}{8mL}. \]

\[ \psi_{1L}^2(x) = \langle x | \psi_{1L}^2(x) \rangle = \sqrt{\frac{1}{L}} \sin \left( \frac{\pi x}{2L} \right), \text{ see part (a) } \]

(d) From the previous question, we’ve already learnt that the probability of measuring the state’s energy \( E_{nL}^2 \) right after the well expands is

\[ |\langle \psi_{nL}^2(x, 0) | \psi_1^L(x, 0) \rangle|^2 = |c_n(0)|^2. \]

Hence

\[ \text{Prob (measure } E_{1L}^2) = |c_1(0)|^2 = \left( \frac{1}{3\pi} \right)^2 = 0.0038. \]

(e)

\[ \text{Prob } (E_{2L}^2) = 0, \quad \text{Prob } (E_{3L}^2) = \left( \frac{4}{\pi(4 - 9)} \right)^2 = \left( \frac{-4}{5\pi} \right)^2 = 0.064 \]
(f) The wavefunctions are shown below

\[ \psi_1, \psi_2, \psi_3 \]

It can be seen that \( \Psi(x,0) \) is a superposition of only \( \psi_1^{2L}, \psi_3^{2L}, \psi_5^{2L}, \ldots \), in the exact proportion, adjusted such that \( \Psi(x,0) \) is zero in the region \([L, 2L] \). This shows that there is zero probability of detecting the energies \( E_2^{2L}, E_4^{2L}, E_6^{2L} \) etc.

(g) 

\[
\Psi(x, t) = \sum_n c_n(0)e^{-iE_n t/\hbar} \psi_n(x) \\
= \frac{4}{\pi} \left[ \sum_{n=1,5,9} \frac{1}{4-n^2} e^{-iE_n t/\hbar} \psi_n(x) - \sum_{n=3,7,11} \frac{1}{4-n^2} e^{-iE_n t/\hbar} \psi_n(x) \right] \\
= \frac{4}{\pi} \sum_{n=0,1,2,\ldots} (-1)^{n+1} \frac{e^{-iE_{2n+1} t/\hbar}}{4-(2n+1)^2} \psi_{2n+1}(x) \\
= \frac{4}{\pi} \sum_{n=0,1,2,\ldots} (-1)^{n+1} \frac{e^{-i(2n+1)^2\pi^2 t/8mL^2\hbar}}{4-(2n+1)^2} \sin \left( \frac{(2n+1)\pi}{2} \right).
\]